EVALUATION GUIDELINES - Course paper

EBA 29101
Mathematics for Business Analytics

Department of Economics

Start date:	04.03.2020	Time 09:00
Finish date:	11.03.2020	Time 12:00

For more information about formalities, see examination paper.

Term paper - EBA2911[1] Mathematics for Business Analytics
4 March – 11 March 2020

SOLUTIONS

Problem 1

a) A product equals zero if and only if one of the factors equals 0. This gives the following solutions: $\underline{\underline{x=0}}$, $\underline{\underline{x=4}}$, $\underline{\underline{x=-2{,}5}}$ and $\underline{\underline{x=\frac{10}{3}}}$.

b) We convert the fractions so that they get a common denominator:

$$\frac{2(x+2)}{(x+1)(x+2)}+\frac{3x(x+1)}{(x+1)(x+2)}=\frac{2(x+1)(x+2)}{(x+1)(x+2)}$$

This gives the equation $2(x+2)+3x(x+1)=2(x+1)(x+2)$. We resolve and simplify: $3x^2+5x+4=2x^2+6x+4$. We subtract the right hand side from both sides and get an equation on standard form: $x^2-x=0$. We have x as a common factor in each of the terms and we can thus put x outside a parenthesis: $x(x-1)=0$. We get the solutions $\underline{\underline{x=0}}$ and $\underline{\underline{x=1}}$.

c) We isolate one of the square roots: $\sqrt{2x+1}=\sqrt{x}+1$ and squaring each side gives $2x+1=x+2\sqrt{x}+1$. We repeat with the remaining square root: $x=2\sqrt{x}$ gives $x^2=4x$, i.e. $x(x-4)=0$ with the roots $x=0$ og $x=4$. Since we have squared each side of the equation twice we might have introduced false solutions. Hence we test the solutions in the original equation:

$$\begin{array}{lllll} \underline{x=0} & \text{left h.s:} & \sqrt{2\cdot 0+1}=1 & \text{right h.s:} & \sqrt{0}+1=1 \quad \text{ok} \\ \underline{x=4} & \text{left h.s:} & \sqrt{2\cdot 4+1}=3 & \text{right h.s:} & \sqrt{4}+1=3 \quad \text{ok} \end{array}$$

Hence $\underline{\underline{x=0}}$ and $\underline{\underline{x=4}}$ are the solutions to the equations.

d) This is a geometric series with first term $a_1=x$, multiplicative factor $k=\frac{x}{1{,}03}$ and the number of terms $n=20$. Hence the equation can be written as

$$x\cdot\frac{\left(\frac{x}{1{,}03}\right)^{20}-1}{\frac{x}{1{,}03}-1}=0$$

The expression is not valid if $x=1{,}03$ (0 in the denominator). This is not a solution of original equation (the left h.s. with $x=1{,}03$ equals $20\cdot 1{,}03$). We get

$$\left(\frac{x}{1{,}03}\right)^{20}=1 \quad \text{i.e.} \quad \frac{x}{1{,}03}=\pm 1 \quad \text{dvs} \quad x=\pm 1{,}03$$

Since $x=1{,}03$ is not valid, the only solutions are $\underline{\underline{x=0}}$ and $\underline{\underline{x=-1{,}03}}$.

Problem 2

a) The inequality is already in standard form: There is a 0 on the right h.s. and the left h.s. is a fraction where the numerator and the denominator is factored as much as possible. Then we can use a sign diagram, see figure 1.

[1] Exam code EBA29101

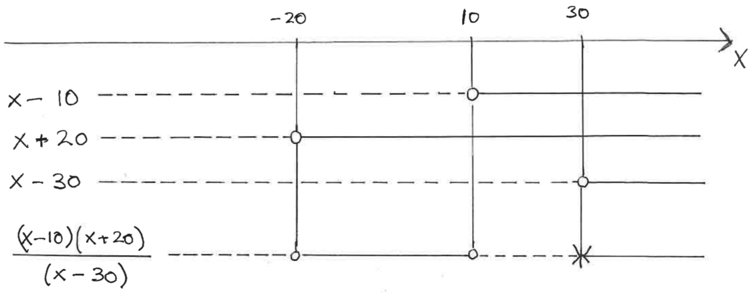

Figure 1: Sign diagram in 2a

We get the solutions $\underline{\underline{x \in [-20\,,\,10] \cup \langle 30\,,\,\rightarrow\rangle}}$. Alternative way of writing: $\underline{\underline{-20 \leqslant x \leqslant 10 \text{ or } x > 30}}$.

b) Here we need to get the inequality on standard form.

$$\frac{x(x-10)}{(x-5)} \geqslant -10 \quad\Leftrightarrow\quad \frac{x(x-10)}{(x-5)} + 10 \geqslant 0 \quad\Leftrightarrow\quad \frac{x(x-10)}{(x-5)} + \frac{10(x-5)}{(x-5)} \geqslant 0$$

$$\Leftrightarrow\quad \frac{x^2-50}{(x-5)} \geqslant 0 \quad\Leftrightarrow\quad \frac{(x+\sqrt{50})(x-\sqrt{50})}{(x-5)} \geqslant 0$$

Then we can use a sign diagram, see figure 2.

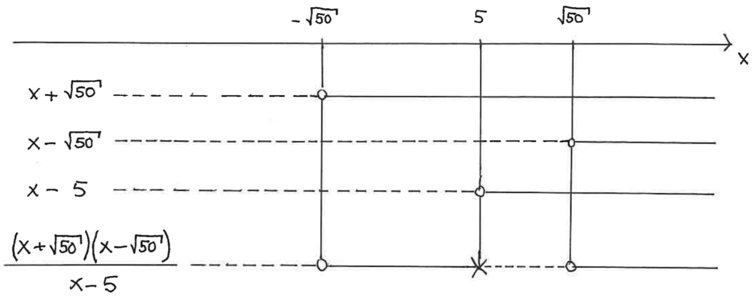

Figure 2: Sign diagram in 2b

We get the solutions $\underline{\underline{x \in [-\sqrt{50}\,,\,5\rangle \cup [\sqrt{50}\,,\,\rightarrow\rangle}}$. Alternative way of writing: $\underline{\underline{-\sqrt{50} \leqslant x < 5 \text{ or } x > \sqrt{50}}}$.

Problem 3

We guess that the polynomial $f(x)$ has an integer solution, which in that case must be a factor in 49, i.e. ±1, ±7, ±49. We test and get that $x = 7$ is a root. Then $x-7$ is a factor in $f(x)$. We use polynomial division to find $f(x) : (x-7)$.

$$\begin{array}{l}
\big(\ \ x^4 - 14x^3 + 50x^2 - 14x + 49\big) : \big(x-7\big) = x^3 - 7x^2 + x - 7 \\
\underline{-x^4 \ \ + 7x^3} \\
\qquad\quad -7x^3 + 50x^2 \\
\qquad\quad \underline{\ \ \ 7x^3 - 49x^2} \\
\qquad\qquad\qquad\ x^2 - 14x \\
\qquad\qquad\quad \underline{-x^2 \ \ + 7x} \\
\qquad\qquad\qquad\qquad -7x + 49 \\
\qquad\qquad\qquad\qquad \underline{\ \ \ 7x - 49} \\
\qquad\qquad\qquad\qquad\qquad\ \ 0
\end{array}$$

Then we seek an integer root in $x^3 - 7x^2 + x - 7$. The possibilities are ±1, ±7 and 7 is a root. Then $x-7$ is a factor and we use polynomial division again.

$$
\begin{array}{l}
\big(\quad x^3-7x^2+x-7\big):\big(x-7\big)=x^2+1 \\
\underline{-x^3+7x^2} \\
\qquad\qquad x-7 \\
\qquad\quad \underline{-x+7} \\
\qquad\qquad\quad 0
\end{array}
$$

Since x^2+1 has no roots we cannot factorise any further. Hence the answer is
$f(x)=x^4-14x^3+50x^2-14x+49=\underline{\underline{(x-7)^2(x^2+1)}}$.

Problem 4

a) The sum of the present values to each of the payments is

$$\frac{200\,000}{1{,}05^6}+\frac{200\,000}{1{,}05^7}+\cdots+\frac{200\,000}{1{,}05^{n+4}}+\frac{200\,000}{1{,}05^{n+5}}$$

b) We read the geometric series backwards. Then the first term is $a_1=\frac{200\,000}{1{,}05^{n+5}}$, the multiplicative factor $k=1{,}05$ and the number of terms is n. Then the sum is

$$\frac{200\,000}{1{,}05^{n+5}}\cdot\frac{1{,}05^n-1}{0{,}05}$$

We can make a table

n	10	20	50
Present value	1 210 036,27	1 952 893,58	2 860 799,06

c) If the cash flow continues forever we get the present value

$$a_1\cdot\frac{1}{1-k}=\frac{200\,000}{1{,}05^6}\cdot\frac{1}{1-\frac{1}{1{,}05}}=\underline{\underline{3\,134\,104{,}67}}$$

Here we read the infinite geometric series from the left and then the multiplicative factor is $k=\frac{1}{1{,}05}$.

Problem 5

a) The future value of costs 8 years from now with 15% interest gives the price of the patent.

$$250\cdot1{,}15+250\cdot1{,}15^2+\cdots+250\cdot1{,}15^7+250\cdot1{,}15^8=250\cdot1{,}15\cdot\frac{1{,}15^8-1}{0{,}15}=3\,946{,}46$$

because $a_1=250\cdot1{,}15$, $k=1{,}15$ and $n=8$. Hence the price of the patent should be $\underline{\underline{3\,946{,}46\text{ million}}}$.

b) If r is the internal rate of return the present value of the cash flow is $\dfrac{6000}{(1+r)^8}-1200$ which should be equal to 0. We get the equation $(1+r)^8=\dfrac{3600}{1200}=3$ with the solution $r=\sqrt[8]{3}-1=\underline{\underline{14{,}72\%}}$.

Problem 6

a) The first payment is $4 \cdot 12 = 48$ terms from now. The term interest is $\frac{3}{12}\% = 0{,}25\%$. Altogether there are $12 \cdot 30 = 360$ terms. This gives the geometric series

$$\frac{10\,000}{1{,}0025^{48}} + \frac{10\,000}{1{,}0025^{49}} + \cdots + \frac{10\,000}{1{,}0025^{406}} + \frac{10\,000}{1{,}0025^{407}}$$

With $a_1 = \frac{10\,000}{1{,}0025^{407}}$, $k = 1{,}0025$ and $n = 360$ the sum is given as

$$\frac{10\,000}{1{,}0025^{407}} \cdot \frac{1{,}0025^{360} - 1}{0{,}0025} = \underline{\underline{2\,109\,256{,}14}}$$

b) We denote the new monthly payment for A. The present value of the cash flow is then the sum of the geometric series

$$\frac{10\,000}{1{,}0025^{48}} + \frac{10\,000}{1{,}0025^{49}} + \cdots + \frac{10\,000}{1{,}0025^{119}} + \frac{A}{1{,}005^{120}} + \frac{A}{1{,}005^{121}} + \cdots + \frac{A}{1{,}005^{407}}$$

which should be equal to the morgage $2\,109\,256{,}14$. We get the equation

$$\frac{10\,000}{1{,}0025^{119}} \cdot \frac{1{,}0025^{72} - 1}{0{,}0025} + \frac{A}{1{,}005^{407}} \cdot \frac{1{,}005^{288} - 1}{0{,}005} = 2\,109\,256{,}14$$

with solution

$$A = \left(2\,109\,256{,}14 - \frac{10\,000}{1{,}0025^{119}} \cdot \frac{1{,}0025^{72} - 1}{0{,}0025}\right) \cdot 1{,}005^{407} \cdot \frac{0{,}005}{1{,}005^{288} - 1} = \underline{\underline{18\,097{,}81}}$$

Problem 7

a) The cash flow looks like this:

Year	3	4	...	13	14
Payment	A	$1{,}1 \cdot A$	...	$1{,}1^{10} \cdot A$	$1{,}1^{11} \cdot A$

It gives the following sum of present values:

$$\frac{A}{1{,}05^3} + \frac{1{,}1 \cdot A}{1{,}05^4} + \cdots + \frac{1{,}1^{10} \cdot A}{1{,}05^{13}} + \frac{1{,}1^{11} A}{1{,}05^{14}}$$

If we read this geometric series from the left we get $a_1 = \frac{A}{1{,}05^3}$, $k = \frac{1{,}1}{1{,}05}$ og $n = 12$.

b) The series then has the sum

$$\frac{A}{1{,}05^3} \cdot \frac{\left(\frac{1{,}1}{1{,}05}\right)^{12} - 1}{\frac{1{,}1}{1{,}05} - 1}$$

For the contract to be balanced this sum has to be equal to 80 million. We get the equation

$$\frac{A}{1{,}05^3} \cdot \frac{\left(\frac{1{,}1}{1{,}05}\right)^{12} - 1}{\frac{1{,}1}{1{,}05} - 1} = 80 \quad \text{with solution} \quad A = 80 \cdot 1{,}05^3 \cdot \frac{\frac{1{,}1}{1{,}05} - 1}{\left(\frac{1{,}1}{1{,}05}\right)^{12} - 1} = \underline{\underline{5{,}90}}$$

Problem 8

a) All hyperbola functions can be written on the standard form $f(x) = d + \frac{a}{x-s}$ for some fixed numbers a, d and s. Then the line $x = s$ is the vertical asymptote and the line $y = d$ the horizontal asymptote. From the graph we see that $s = 6$ and $d = 10{,}5$. (E.g. consider the line through the points $(5, 11)$ and $(7, 10)$ on the graph. The intersection point with the vertical asymptote is equally far from both points. Then they are equally far from the intersection point with the horizontal asymptote which has to be the same point.) Hence $f(x) = 10{,}5 + \frac{a}{x-6}$. Because $(7, 10)$ is contained in the graph, $f(7) = 10$, i.e. $10{,}5 + \frac{a}{7-6} = 10$. Then we get that $a = -0{,}5$ and $f(x) = \underline{\underline{10{,}5 - \frac{0{,}5}{x-6}}}$.

b) The standard form of the equation of a 'straight' ellipse is

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

where (x_0, y_0) is the centre of the ellipse, a is the horizontal half axis and b is the vertical half axis. From the figure we see that $(x_0, y_0) = (5, 4)$, $a = 9-5 = 4$ and $b = 7-4 = 3$. Hence the equation for the ellipse is

$$\underline{\underline{\frac{(x-5)^2}{16} + \frac{(y-4)^2}{9} = 1}}$$

Problem 9

All second degree functions $f(x)$ can be written on the standard form $f(x) = a(x-s)^2 + d$ where the vertical line $x = s$ is the symmetry axis of the parabola and d is the minimum or maximum of $f(x)$. From the graph we see that $s = 12$ and $d = 15$. It gives $f(x) = a(x-12)^2 + 15$. From $f(7) = 10$ we get the equation $a(7-12)^2 + 15 = 10$, i.e. $a = -0{,}2$. Hence $f(x) = \underline{-0{,}2(x-12)^2 + 15}$.

a) The symmetry axis is the line $x = 12$, so $f(17) = f(7) = \underline{\underline{10}}$ (look at the graph).

b) We solve the equation $-0{,}2(x-12)^2 + 15 = 0$ which gives $(x-12)^2 = \frac{-15}{-0{,}2} = 75$, i.e. $x - 12 = \pm\sqrt{75} = \pm 5\sqrt{3}$ which gives the roots $x = \underline{\underline{12 \pm 5\sqrt{3}}}$.

Problem 10

a) If $ax^2 + bx + c$ is supposed to have roots r_1 and r_2 we get that $ax^2 + bx + c = a(x-r_1)(x-r_2) = ax^2 - a(r_1+r_2)x + ar_1r_2$. Then

$$\begin{cases} b = -a(r_1 + r_2) \\ c = ar_1r_2 \end{cases}$$

Here $r_1 = 2 - \sqrt{2}$, $r_2 = 2 + \sqrt{2}$ and $c = 12$. Then $r_1 + r_2 = 4$ and $r_1r_2 = 4 - 2 = 2$. It gives the equations

$$\begin{cases} b = -4a \\ 12 = 2a \end{cases}$$

The second equation gives $a = 6$ and the the first gives $b = -24$. Hence the expression is $\underline{\underline{6x^2 - 24x + 12}}$.

b) In this case we have $b = -7$, $r_1 + r_2 = 7$ and $r_1r_2 = 10$. This gives the equations

$$\begin{cases} -7 = -7a \\ c = 10a \end{cases}$$

The first equation gives $a = 1$ and then the second gives $c = 10$. Hence the expression is $\underline{\underline{x^2 - 7x + 10}}$.

c) Since it is a second degree expression which only has one root, this root has to give the symmetry axis of the parabola $x = s = 4$. Apart from $x = 4$ the parabola must be entirely on one side of the x-axis. Since $(7, 1)$ is contained in the graph, the graph is turning upwards and $x = 4$ is a minimum point. Hence $d = 0$ and we get the expression $a(x-4)^2$. Since the point $(7, 1)$ is contained in the graph we get the equation $a(7-4)^2 = 1$ which yields $a = \frac{1}{9}$. Hence the expression is $\underline{\underline{\frac{1}{9}(x-4)^2}}$.

Problem 11

Suppose t denotes the middle root and d the distance from the middle one to the smallest. Then the three roots in ascending order is $t-d$, t and $t+2d$ (we suppose that $d > 0$). The the third degree expression is

$$(x-t+d)(x-t)(x-t-2d) = \underline{\underline{x^3-(3t+d)x^2+(3t^2+2dt-2d^2)x-t(t^2+dt-2d^2)}}$$

Problem 12

The profit function $P(x) = R(x) - C(x) = ax - (7200 + 3x) = (a-3)x - 7200$ (for $x \geqslant 0$) has one root $x = \frac{7200}{a-3}$ for $a \neq 3$. We solve the equation $\frac{7200}{a-3} = 150$ and get $a = \frac{7200}{150} + 3 = 51$. For $a > 3$, $P(x)$ is an increasing function and the profit changes from negative to positive at $x = 150$ when $\underline{\underline{a = 51}}$.