EVALUATION GUIDELINES - Course paper

EBA 29101
Mathematics for Business Analytics

Department of Economics

Start date:	05.03.2021	Time 09:00
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For more information about formalities, see examination paper.

Term paper - EBA2911[1] Mathematics for Business Analytics
5 March – 12 March 2021

SOLUTIONS

Problem 1

a) The sum is a geometric series which, read from the right to the left, has $a_1 = 5\,000 \cdot 1.002$, growth factor $k = 1.002$ and number of terms $n = 60$. Then the formula for the sum of a geometric series gives

$$5\,000 \cdot 1.002 \cdot \frac{1.002^{60} - 1}{0.002} = \underline{\underline{319\,041.16}}$$

If you deposit 5 000 every month into an account (with first payment today) giving 2.4% nominal interest, monthly compounding and 60 terms, the sum is the account balance 5 years from now.

b) The sum is a geometric series which, read from the right to the left, has $a_1 = \frac{5\,000}{e^{0.12}}$, growth factor $k = e^{0.002}$ and number of terms $n = 60$. Then the formula for the sum of a geometric series gives

$$\frac{5\,000}{e^{0.12}} \cdot \frac{e^{60 \cdot 0.002} - 1}{e^{0.002} - 1} = \frac{5\,000}{e^{0.12}} \cdot \frac{e^{0.12} - 1}{e^{0.002} - 1} = \underline{\underline{282\,416.30}}$$

If 5000 is the monthly annuity with first payment one month from now, with 2.4% nominal interest and continuous compounding and 5 years down payment period, the given sum is the total amount borrowed (the present value of the cash flow).

Problem 2

a) The total present value of the cash flow is the sum of the present values of the payments:

$$-18 - \frac{25}{1.12} - \frac{15}{1.12^2} + \frac{95}{1.12^8} = \underline{\underline{-13.91}}$$

b) The future value of the cash flow after 7 years is (present value) $\cdot\, 1.12^7 = \underline{\underline{-30.75}}$.

c) One extra payment of $\underline{\underline{+30.75}}$ after 7 years changes the future value of the cash flow after 7 years to 0. Hence the present value of the new cash flow is also 0 and the discount rate of 12% is also the internal rate of return of the cash flow.

Problem 3

a) The present value of a payment of 30 million 7 years from now with nominal interest r (and continuous compounding) is $\frac{30}{e^{7r}}$ which is supposed to be 15 million. This gives the equation $\frac{30}{e^{7r}} = 15$ which simplifies to $2 = \frac{30}{15} = e^{7r}$, that is $7r = \ln(2)$ and $\underline{\underline{r = \frac{\ln 2}{7} = 9.90\%}}$.

b) The same calculation with 10 years gives $R = \frac{\ln 2}{10}$. Hence

$$\frac{R}{r} = \frac{\left(\frac{\ln 2}{10}\right)}{\left(\frac{\ln 2}{7}\right)} = \frac{\left(\frac{1}{10}\right)}{\left(\frac{1}{7}\right)} = \frac{7}{10} = 0.7$$

and so $R = 0.7r$. Alternatively we can start with the equation $\frac{30}{e^{7r}} = \frac{30}{e^{10R}}$ which gives $7r = 10R$ and then $R = 0.7r$.

Problem 4

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a) We square both sides of $\sqrt{2x+3} = x-6$ and get the quadratic equation $2x+3 = x^2-12x+36$, that is $x^2-14x = -33$. Completion of the square gives $(x-7)^2 = 49-33 = 16$ and $x = 7\pm4$ that is $x=3$ or $x=11$. But because we have squared both sides of the equation we have to check is these really are solutions of the given equation.

$\underline{x=3}$: LHS $= \sqrt{2\cdot3+3} = 3$, while RHS $= 3-6 = -3$ hence $x=3$ is not a solution to the given equation.

$\underline{x=11}$: LHS $= \sqrt{2\cdot11+3} = 5$ og RHS $= 11-6 = 5$ hence $\underline{\underline{x=11}}$ is the only solution of the given equation.

b) A product equals 0 if and only if one of the factors equals 0. Then either $\ln(x)-3 = 0$, that is $x = e^3$, or $x^2-400 = 0$, that is $x = \pm20$. Because $e^x > 0$ for all x, $e^x+3 = 0$ has no solutions. Because $\ln(x)$ only is defined for $x > 0$, $\underline{\underline{x = e^3 \text{ and } x = 20}}$ are the only true solutions to the given equation.

c) We multiply each side of the equation $\dfrac{\ln(x)}{\ln(x)-10} = 11$ with $(\ln(x)-10)$ and get the equation $\ln(x) = 11\ln(x)-110$, that is $110 = 10\ln(x)$. Dividing each side by 10 gives the equation $\ln(x) = 11$, so $\underline{\underline{x = e^{11}}}$.

d) We substitute $u = x^{-3}$ into the equation and get $u^2-6u = 16$. Completing the square $(u-3)^2 = 9+16 = 25$ gives $u = 3\pm5$, that is $u = -2$ or $u = 8$. Because $x = \frac{1}{\sqrt[3]{u}}$, the solutions are $\underline{\underline{x = -\frac{1}{\sqrt[3]{2}},\ x = \frac{1}{2}}}$.

Problem 5

Because $e^x > 0$ for all x we see that the left hand side is greater than 0 no matter what x is. Moreover, the denominator is 1 more that the numerator and thus the left hand side is always less than 1. Hence $0 < t < 1$. In fact, for all such values of t the equation has a solution: We multiply the equation with e^x+1 on both sides and get $e^x = te^x + t$ which gives $(1-t)e^x = t$ that is $e^x = \frac{t}{1-t}$ which is ok since $t < 1$. This gives the solution $x = \ln\left(\frac{t}{1-t}\right)$ because the fraction is positive for $0 < t < 1$. This reasoning gives the answers:

a) The equation has solutions for $\underline{\underline{\text{all } t \text{ with } 0 < t < 1 \text{ and no other values}}}$.

b) The solution is $\underline{\underline{x = \ln\left(\frac{t}{1-t}\right) \text{ for } 0 < t < 1}}$.

Problem 6

a) We factorise the numerator: $\frac{x(2-x)}{x-5} \leqslant 0$. With sign diagram we get $\underline{\underline{0 \leqslant x \leqslant 2 \text{ or } x > 5}}$.

b) Subtracting 1 from each side and making the left hand side into one factorised fraction gives $\dfrac{(x+1)(x-3)}{(x+3)(x-4)} \geqslant 0$. With sign diagram we get $\underline{\underline{x < -3 \text{ or } -1 \leqslant x \leqslant 3 \text{ or } x > 4}}$.

c) The inequality is only defined for $x > -4$. Since e^x is a strictly increasing function we can insert the left and the right hand side into $e^{(-)}$ and obtain an equivalent inequality: $5x+20 \leqslant e^3$ which gives $x \leqslant \frac{e^3}{5}-4$. The answer is thus $\underline{\underline{-4 < x \leqslant \frac{e^3}{5}-4}}$.

d) We multiply both sides of the inequality with the positive number $e^{0.1x}$ and get $e^{0.4x} \leqslant 170$. Since $\ln(x)$ is a strictly increasing function we can insert the left and the right hand side into $\ln(-)$ and obtain an equivalent inequality: $0.4x \leqslant \ln(170)$, that is $\underline{\underline{x \leqslant 2.5\cdot\ln(170)}}$.

e) We note that $\ln(5-x)$ is only defined for $x < 5$ and equals 0 if $x = 4$. Moreover, $e^x > 0$ for all x and $4-x^2 = (2-x)(2+x)$. Hence a sign diagram with the factors $\ln(5-x)$, $2-x$ and $2+x$ gives $\underline{\underline{-2 \leqslant x \leqslant 2 \text{ or } 4 \leqslant x < 5}}$.

Problem 7

a) Polynomial division gives:

$$\begin{array}{l} \big(\ 3x^3 \ \ -7x^2 - 10x + 14\big) : \big(x^2 - 4x + 3\big) = 3x + 5 + \dfrac{x-1}{x^2-4x+3} \\ \underline{-3x^3 + 12x^2 \ \ -9x} \\ \qquad\quad 5x^2 - 19x + 14 \\ \qquad \underline{-5x^2 + 20x - 15} \\ \qquad\qquad\qquad x \ \ -1 \end{array}$$

The remainder is $\underline{\underline{x-1}}$.

b) We have

$$\frac{f(x)}{g(x)} = 3x + 5 + \frac{x-1}{(x-1)(x-3)} = 3x + 5 + \frac{1}{x-3}$$

This gives the line $\underline{\underline{x = 3}}$ as a vertical asymptote and the line $\underline{\underline{y = 3x + 5}}$ as a non-vertical asymptote for $\dfrac{f(x)}{g(x)}$.

Problem 8

a) We solve the equation $y = -0.2x + 20$ for x and get $x = -5y + 100$, that is $\underline{\underline{g(x) = -5x + 100}}$. We have $f(0) = 20$ and $f(10) = 18$ and $f(x)$ is strictly decreasing in the whole domain $D_f = [0, 10]$. Hence $\underline{\underline{D_g = V_f = [18, 20]}}$ and $\underline{\underline{V_g = D_f = [0, 10]}}$.

b) We solve the equation $y = e^{-0.1x} + 3$ for x and get $x = -10\ln(y-3)$, that is $\underline{\underline{g(x) = -10\ln(x-3)}}$. We have that $f(0) = 4$, $f(x)$ is strictly decreasing in the whole domain and the line $y = 3$ is a horizontal asymptote for $f(x)$. At the same time $f(x) > 3$ for all x. Hence $\underline{\underline{D_g = V_f = \langle 3, 4]}}$ and $\underline{\underline{V_g = D_f = [0, \rightarrow\rangle}}$.

Problem 9

All second degree polynomial functions can be written as $f(x) = a(x-s)^2 + d$ where $x = s$ is the symmetry line and d is the maximum value $f(s)$ if $a < 0$. Hence we get $\underline{\underline{f(x) = a(x-70)^2 + 200 \text{ for a parameter } a < 0}}$.

Problem 10

All hypebola functions can be written as $f(x) = c + \frac{a}{x-b}$ where $x = b$ is the vertical asymptote and $y = c$ is the horizontal asymptote. Hence $f(x) = 11 + \frac{a}{x-9}$ and $f(4) = 11 + \frac{a}{4-9} = 11 - 0.2a$ which is given to be 12. Solving the equation $11 - 0.2a = 12$ for a we get $a = -5$. That is $f(x) = 11 - \frac{5}{x-9}$. The graph intersects the y-axis in the point $(0, f(0)) = \underline{\underline{(0, \frac{104}{9})}}$ and the x-axis in the point $\underline{\underline{(\frac{104}{11}, 0)}}$. Here $x = \frac{104}{11}$ is the solution of the equation $f(x) = 0$.

Problem 11

From the standard ellipse equation we get

$$\frac{(x-5)^2}{9}+\frac{(y-6)^2}{16}=1$$

The line has equation $x+y=10$ which gives $y=10-x$. Substituting for y in the ellipse equation we get

$$\frac{(x-5)^2}{9}+\frac{(4-x)^2}{16}=1$$

which gives

$$25x^2-232x=-400$$

which has solutions

$$x=\frac{116\pm 24\sqrt{6}}{25}$$

Inserted into $y=10-x$ we get

$$y=10-\left(\frac{116\pm 24\sqrt{6}}{25}\right)=\frac{134\mp 24\sqrt{6}}{25}$$

which gives the points

$$\underline{\underline{\left(\frac{116-24\sqrt{6}}{25},\frac{134+24\sqrt{6}}{25}\right)}} \quad \text{and} \quad \underline{\underline{\left(\frac{116+24\sqrt{6}}{25},\frac{134-24\sqrt{6}}{25}\right)}}$$