Course paper 1 - EBA2910[1] - Mathematics for Business Analytics

4 – 11 March 2022

SOLUTIONS

Problem 1

a) $\ln(x) = \frac{a}{5} = 0{,}2a$ inserted into e^x gives $\underline{\underline{x = e^{0{,}2a}}}$

b) $e^x = 3(a-1)$ inserted into $\ln(x)$ gives $\underline{\underline{x = \ln(a-1) + \ln(3) \text{ for } a > 1 \text{ and no solutions for } a \leqslant 1}}$

c) Common fraction: $\frac{a+(x-3)(2-5a)}{x-3} = 0$ i.e. $\frac{(2-5a)x+16a-6}{x-3} = 0$ i.e. $(2-5a)x + 16a - 6 = 0$, i.e. $\underline{\underline{x = \frac{16a-6}{5a-2} \text{ for } a \neq 0{,}4 \text{ and no solutions for } a = 0{,}4}}$

d) $x^4 = \frac{1}{-a}$ i.e. $\underline{\underline{x = \pm\frac{1}{\sqrt[4]{-a}} \text{ for } a < 0 \text{ and no solutions for } a \geqslant 0}}$

Problem 2

a) $\ln(x-a) \leqslant 5$ inserted into the strictly increasing e^x gives $0 < x - a \leqslant e^5$, i.e. $\underline{\underline{a < x \leqslant a + e^5}}$

b) Factorisation gives $(a+1)x \geqslant 1$. When dividing both sides with $a+1$ we have to consider the sign: $\underline{\underline{x \geqslant \frac{1}{a+1} \text{ for } a > -1}}$, $\underline{\underline{x \leqslant \frac{1}{a+1} \text{ for } a < -1}}$ and $\underline{\underline{\text{no solutions for } a = -1}}$

c) Common fraction $\frac{x-a+2(x-3)}{x-3} \geqslant 0$ i.e. $\frac{3x-a-6}{x-3} \geqslant 0$. The denominator is increasing and is zero for $x = 3$, the numerator is also increasing and is zero for $x = \frac{a+6}{3}$. Then we can make a sign diagram. There are three cases.

   i) $\frac{a+6}{3} < 3$, i.e. $a < 3$: then the solutions of the inequality are $\underline{\underline{x < \frac{a+6}{3} \text{ or } x > 3}}$.

   ii) $\frac{a+6}{3} > 3$, i.e. $a > 3$: then the solutions of the inequality are $\underline{\underline{x < 3 \text{ or } x > \frac{a+6}{3}}}$.

   iii) $\frac{a+6}{3} = 3$, i.e. $a = 3$: then the solutions of the inequality are $\underline{\underline{\text{all } x \neq 3}}$.

d) $e^{x-a} \leqslant a$ has no solutions for $a \leqslant 0$. If $a > 0$ we insert both sides into the strictly increasing $\ln(x)$ and get the inequality $x - a \leqslant \ln(a)$, i.e. $\underline{\underline{x \leqslant a + \ln(a)}}$

Problem 3

(a) The total present value is

$$-60 - \frac{60}{1{,}15^2} + \frac{80}{1{,}15^8} + \frac{80}{1{,}15^9} + \frac{80}{1{,}15^{10}} = \underline{\underline{-36{,}70}}$$

(b) The future value of the cash flow after 7 years is

$$-60 \cdot 1{,}15^7 - 60 \cdot 1{,}15^5 + \frac{80}{1{,}15} + \frac{80}{1{,}15^2} + \frac{80}{1{,}15^3} = \underline{\underline{-97{,}62}}$$

Note that this is the same as the total present value multiplied with the growth factor for 7 years: $-36{,}70 \cdot 1{,}15^7 = -97{,}62$.

(c) The future value of the cash flow after 7 years is $-97{,}62$ hence if Kåre adds a payment of $\underline{\underline{\text{97,62 seven years from now}}}$ the future value of the cash flow after 7 years is 0. This implies that the internal rate of return of the new cash flow is 15% because the total present value is of the new cash flow is $\frac{0}{1{,}15^7} = 0$.

[1] Exam code EBA29101

Problem 4

(a) The present value is $\frac{60\,\text{mill}}{1{,}09^7} = \underline{\underline{32{,}82}}$ million.

(b) The present value 32,82 million is what you have to deposit today for the balance to be 60 million 7 years from now if the interest is 9%.

(c) The present value of 60 million 6 years from now with interest r is $\frac{60\,\text{mill}}{(1+r)^6}$. We get the equation $\frac{60\,\text{mill}}{1{,}09^7} = \frac{60\,\text{mill}}{(1+r)^6}$. It gives $(1+r)^6 = 1{,}09^7$, i.e. $1+r = (1{,}09^7)^{\frac{1}{6}}$, i.e. $r = 1{,}09^{\frac{7}{6}} - 1 = \underline{\underline{10{,}58\%}}$.

(d) See the calculation in (c).

Problem 5

The polynomial
$g(x) = (x+2)(x-(2-\sqrt{5}))(x-(2+\sqrt{5})) = (x-2)(x^2-4x-1) = x^3-2x^2-9x-2$ has the given roots, but the constant term is -2. We therefore multiply $g(x)$ with -2 and get $\underline{\underline{f(x) = -2x^3+4x^2+18x+4}}$ which has the same roots.

Problem 6

The quadratic equation has solutions exactly if the number under the root in the abc-formula is bigger or equal to 0, i.e. $(2a)^2 - 4\cdot 1\cdot 5 \geqslant 0$ i.e. $a^2 \geqslant 5$ i.e. $\underline{\underline{|a| \geqslant \sqrt{5}}}$

Problem 7

We do polynomial division and get the remainder $a^2-6a+13$ which is given as 5. Hence we solve the equation $a^2-6a+13 = 5$ and get $\underline{\underline{a = 2 \text{ or } a = 4}}$.

Problem 8

We let $x = t$ be the middle root. Then the two other roots are $x = t-1$ and $x = t+1$. This gives the third degree polynomial with these roots as
$(x-(t+1))(x-(t-1))(x-t) = (x^2-2tx+t^2-1)(x-t) = \underline{\underline{x^3-3tx^2+(3t^2-1)x-t(t^2-1)}}$.

Problem 9

a) In the standardform $f(x) = c + \frac{a}{x-b}$, $x = b = 6$ is the vertical asymptote while $y = c = 10{,}5$ is the horizontal asymptote. Hence $f(x) = 10{,}5 + \frac{a}{x-6}$. Moreover, $f(7) = 10{,}5 + \frac{a}{7-6} = 10$ i.e. $a = -0{,}5$ and $\underline{\underline{f(x) = 10{,}5 - \frac{0{,}5}{x-6}}}$.

b) The standard form for the equation of a «straight» ellipse is

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

where (x_0, y_0) is the centre of the ellipse, a is the horizontal semi-axis and b is the vertical semi-axis. The symmetry lines passes through the centre of the ellipse so $(x_0, y_0) = (5, 4)$ and the equation is hence

$$\frac{(x-5)^2}{a^2} + \frac{(y-4)^2}{b^2} = 1$$

Because $(9, 4)$ is contained in E we get the equation

$$\frac{(9-5)^2}{a^2} + \frac{(4-4)^2}{b^2} = 1 \quad \text{i.e.} \quad \frac{16}{a^2} = 1$$

i.e. $a = 4$. Correspondingly, from $(5, 7)$ we get the equation

$$\frac{(5-5)^2}{a^2} + \frac{(7-4)^2}{b^2} = 1 \quad \text{i.e.} \quad \frac{9}{b^2} = 1$$

i.e. $b = 3$. Hence

$$\underline{\underline{\frac{(x-5)^2}{16} + \frac{(y-4)^2}{9} = 1}}$$

Problem 10

The standard form $f(x) = a(x-s)^2 + d$ has $x = s$ as symmetry line. We see that $x = 7$ and $x = 17$ is equally far from the symmetry line since $f(7) = f(17)$. Hence $s = \frac{7+17}{2} = 12$. We also have $d = 15$ so $f(x) = a(x-12)^2 + 15$. From $f(17) = 10$ we get the equation $a(17-12)^2 + 15 = 10$, i.e. $25a = -5$, i.e. $a = -0{,}2$ and $\underline{f(x) = -0{,}2(x-12)^2 + 15}$. The we get

a) $f(9) = -0{,}2(9-12)^2 + 15 = -1{,}8 + 15 = \underline{\underline{13{,}2}}$.

b) Solving the equation $-0{,}2(x-12)^2 + 15 = 0$ gives $(x-12)^2 = (-5)(-15) = 75$, i.e. $\underline{\underline{x = 12 \pm 5\sqrt{3}}}$.

Problem 11

a) We put $y = f(x)$ and solve for x. I.e. $y = -\frac{x}{3} + 7$ which gives $x = -3y + 21$. Then the expression for the inverse function is $g(x) = \underline{\underline{-3x + 21}}$. The domain of definition D_g always equals the range of $f(x)$. Because $f(x)$ is a decreasing function the largest value is $f(0) = 7$ and the smallest value is $f(21) = 0$. Hence $D_g = R_f = \underline{\underline{[0, 7]}}$. Finally, the range of $g(x)$ is always equal to the domain of $f(x)$, i.e. $R_g = \underline{\underline{[0, 21]}}$.

b) We put $y = \ln(x+2) - 5$ and solve for x. We add 5 on each side. It gives $\ln(x+2) = y + 5$. Inserting both sides into e^x we get

$$e^{\ln(x+2)} = e^{y+5}, \quad \text{i.e.} \quad x + 2 = e^{y+5}, \quad \text{i.e.} \quad x = e^{y+5} - 2$$

Hence the expression for the inverse function is

$$g(x) = \underline{\underline{e^{x+5} - 2}}$$

Because $f(x)$ is a strictly increasing function which approaches $-\infty$ when x approaches -2 from above and $f(x)$ increases without bounds when x approaches $+\infty$, the range R_f = the whole number line and so $D_g = \underline{\underline{\text{the whole number line}}}$. Moreover $R_g = D_f = \underline{\underline{\langle -2, \infty \rangle}}$.