EVALUATION GUIDELINES - Take-home examination

EBA 29102
Mathematics for Business Analytics

Department of Economics

Start date:	10.12.2020	Time 14:00
Finish date:	10.12.2020	Time 17:00

For more information about formalities, see examination paper.

Home exam in EBA2910[1] - Mathematics for Business Analytics

10 December 2020

SOLUTIONS

Oppgave 1

$$
\begin{array}{l}
\left(\ \ 6x^3+23x^2+19x+14\right):\left(x+3\right)=6x^2+5x+4+\dfrac{2}{x+3} \\
\underline{-6x^3-18x^2} \\
\qquad\quad 5x^2+19x \\
\qquad \underline{-5x^2-15x} \\
\qquad\qquad\qquad 4x+14 \\
\qquad\qquad\quad \underline{-4x-12} \\
\qquad\qquad\qquad\qquad\ 2
\end{array}
$$

The remainder is hence $\underline{\underline{2}}$.

Problem 2

If r is the internal rate of return the total present value of the cash flow is $-25+\frac{50}{e^{5r}}=0$. This gives the equation $e^{5r}=\frac{50}{25}=2$, that is $r=\underline{\underline{\frac{\ln 2}{5}}}=\underline{\underline{13{,}86\%}}$.

Problem 3

By the chain rule with $u(p)=-0{,}2p$ we get $D'(p)=-0{,}2\cdot D(p)$ and hence the elasticity function is

$$\varepsilon(p)=\frac{D'(p)\cdot p}{D(p)}=\frac{-0{,}2\cdot D(p)\cdot p}{D(p)}=-0{,}2p$$

The demand is *elastic* for those prices p such that $\varepsilon(p)<-1$, that is $-0{,}2p<-1$, that is $\underline{\underline{p>5}}$.

The demand is *inelastic* for those prices p such that $\varepsilon(p)>-1$, that is $\underline{\underline{0<p<5}}$.

The demand is *unit elastic* for those prices p such that $\varepsilon(p)=-1$, that is $\underline{\underline{p=5}}$.

Problem 4

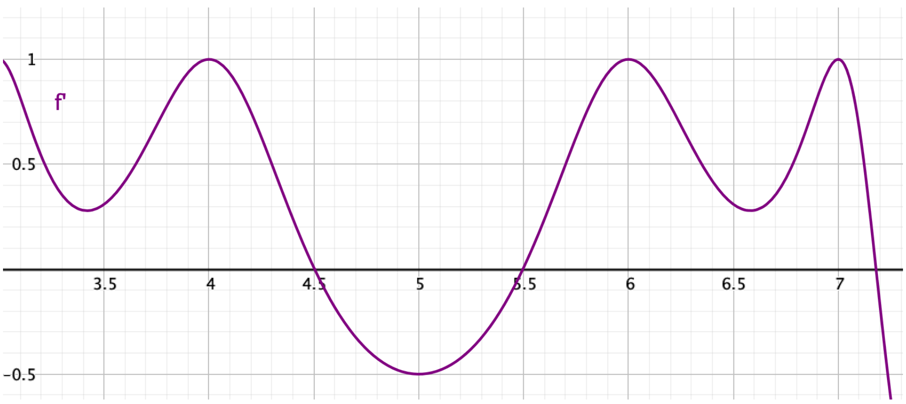

Figure 1: The graph of $f'(x)$

[1] Exam code EBA29102

(A) $f(x)$ *has three stationary points* is correct because a zero (root) for $f'(x)$ is a stationary point for $f(x)$ and we count three zeros for $f'(x)$.
(B) $f(x)$ *is convex in the interval* $[5, 6]$ is correct because $f(x)$ is convex for those x such that $f''(x) \geqslant 0$ which is ok since we can see that the slope of the tangent of the graph of $f'(x)$, which is $f''(x)$, is greater or equal to 0 in the interval.
(C) $f(x)$ *is convex in the interval* $[4{,}5\ ,\ 5{,}5]$ is not correct because the slope of the tangent of the graph of $f'(x)$, which is $f''(x)$, is negative for x between 4,5 and 5.
(D) $f(x)$ *is increasing in the interval* $[6, 7]$ is correct because $f'(x)$ is positive for all x in the interval.

Problem 5

If a product of three numbers equals 0 at least one of the factors must be 0. That is $x^2 - 5 = 0$ which gives $x = \pm\sqrt{5}$, or $e^x - 2 = 0$ which gives $x = \ln(2)$, or $\ln(x) = 0$ which gives $x = e^0 = 1$. But $\ln(x)$ is only defined for positive x and then the same applies to the whole equation. Hence x is either $\underline{\underline{\sqrt{5}}}$, $\underline{\underline{\ln(2)}}$, or $\underline{\underline{1}}$.

Problem 6

If you every month for 10 years pay 5000 into an account earning 0,2% interest every month (that is 2,4% nominal interest and monthly compounding) with the first payment today, the sum represents the future value of the cash flow (that is account balance) 10 years from now. The monthly growth factor is $1 + 0{,}2\% = 1{,}002$ and $5000 \cdot 1{,}002^n$ represents the future value of payment no. $(121 - n)$.

Problem 7

To find the function expression $g(x)$ we first solve the equation $y = \ln(x-1)$ for x. To get $x-1$ «out» of ln we insert the left hand side and the right hand side of the equation into $e^{(-)}$. It gives $e^y = e^{\ln(x-1)} = x - 1$, that is $x = e^y + 1$. Then we change variables and get $\underline{\underline{g(x) = e^x + 1}}$.

Bescause $f(x) = \ln(x-1)$ is (strictly) increasing the smallest value is $f(2) = \ln(1) = 0$. Because $f(x)$ increases without any upper bound ($\lim_{x\to\infty} \ln(x-1) = \infty$) and is continuous we get $R_f = [0, \infty\rangle$. (Alternative argument: the equation $f(x) = c$ has a solution x in the interval $D_f = [2, \infty\rangle$ for any $c \geqslant 0$, namely $x = e^c + 1$). In general $D_g = R_f$. Hence $\underline{\underline{D_g = [0, \infty\rangle}}$.

Problem 8

The hyperbola function can be written on standard form as $f(x) = c + \frac{a}{x-b}$ where $b = 5$ and $c = 100$, that is $f(x) = 100 + \frac{a}{x-5}$. From $f(6) = 112$ we hence get the equation $100 + \frac{a}{6-5} = 112$, that is $a = 12$ and $f(x) = 100 + \frac{12}{x-5}$. Then $f(17) = 100 + \frac{12}{17-5} = \underline{\underline{101}}$.

Problem 9

The chain rule for differentiation of composed functions says that $f'(x) = g'(u(x)) \cdot u'(x)$. This gives $f'(12) = g'(u(12)) \cdot u'(12) = g'(52) \cdot u'(12) = 32 \cdot (-0{,}1) = \underline{\underline{-3{,}2}}$.

Problem 10

The second degree Taylor polynomial for $f(x)$ in 30 is $P_2(x) = 700 + 5(x-30) - 0{,}5(x-30)^2$. Then $f(31) \approx P_2(31) = \underline{\underline{704{,}5}}$.

Problem 11

In figure 2 you see the graph of the cost function with the tangent through the origin and a non-optimal point Q.

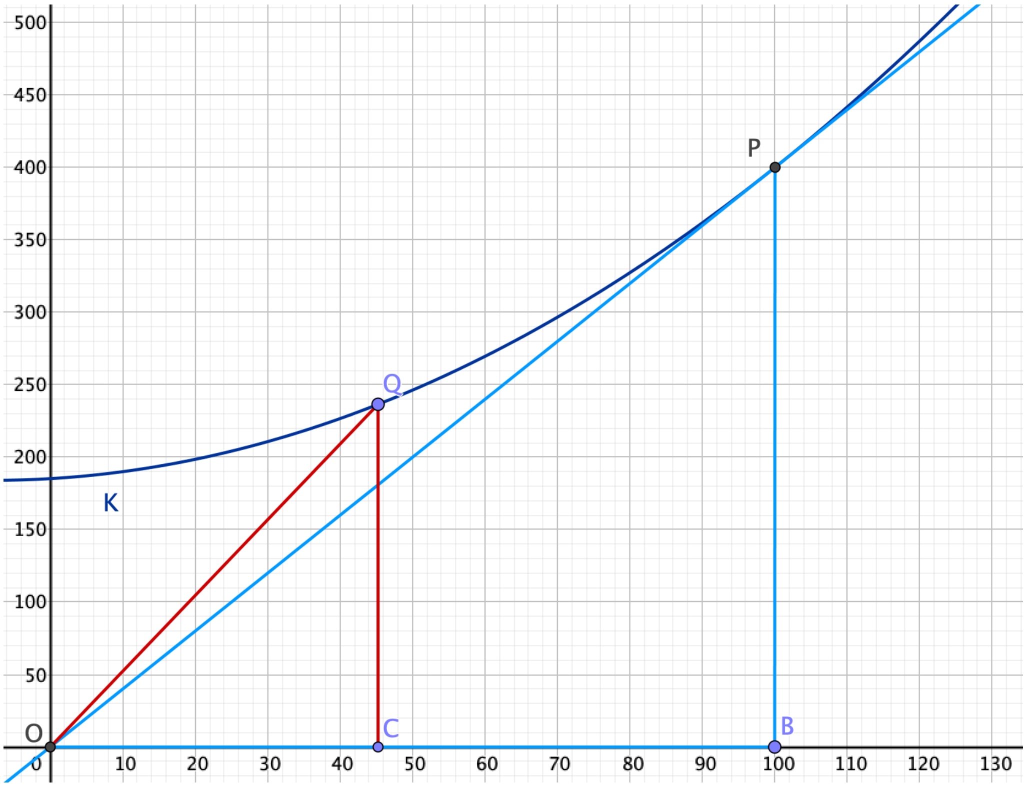

Figure 2: The cost function $K(x)$

When the cost function is strictly convex ($K''(x) > 0$) as it is here, the tangent of the cost function which passes through the origin will have the minimal unit cost as slope and the x-coordinate of the tangent point will be the cost optimum. We note that $P = (100, 400)$ approximately is this tangent point, therefore the minimal unit cost equals $\frac{400}{100} = \underline{\underline{4}}$ and the cost optimum is $\underline{\underline{x = 100}}$.

By a result in the course cost optimum is the solution of the equation $K'(x) = A(x)$ where $A(x) = \frac{K(x)}{x}$ is the average unit cost for x produced units. The tangent method for finding the minimal unit cost hence works because the slope of the tangent passing through the origin is both equal to $K'(100)$ and $\frac{K(100)}{100} = A(100)$. We see that the line through the origin and another point Q on the graph of $K(x)$ has higher slope. Hence $x = 100$ is cost optimum.

Problem 12

The standard form of the ellipse equation is

$$\frac{(x-3)^2}{(\sqrt{3})^2} + \frac{(y-4)^2}{(\sqrt{6})^2} = 1$$

By multiplying both sides of the equation with $(\sqrt{3})^2 \cdot (\sqrt{6})^2 = 3 \cdot 6 = 18$ we get the equation

$$6(x-3)^2 + 3(y-4)^2 = 18 \qquad \text{that is} \qquad 6x^2 - 36x + 54 + 3y^2 - 24y + 48 = 18$$

We differentiate both sides of the equation with respect to x and get (use the chain rule)

$$12x - 36 + 6yy' - 24y' = 0$$

which we solve for y' and obtain $6(y-4)y' = 12(3-x)$, that is

$$y' = \frac{12(3-x)}{6(y-4)} = \frac{2(3-x)}{(y-4)}$$

Inserting $x=2$ into the ellipse equation gives $\frac{(2-3)^2}{3} + \frac{(y-4)^2}{6} = 1$, that $(y-4)^2 = 6(1-\frac{1}{3}) = 6 \cdot \frac{2}{3} = 4$. Hence $y = 4 \pm 2$, that is $y = 6$ or $y = 2$.
For the point $(2,6)$ the slope of the tangent function $h_1(x)$ is $y' = \frac{2(3-2)}{(6-4)} = 1$. The slope-point formula then gives

$$h_1(x) - 6 = 1 \cdot (x-2) \qquad \text{that is} \qquad \underline{\underline{h_1(x) = x+4}}$$

For the point $(2,2)$ the slope of the tangent function $h_2(x)$ is $y' = \frac{2(3-2)}{(2-4)} = -1$. The slope-point formula then gives

$$h_2(x) - 2 = (-1) \cdot (x-2) \qquad \text{that is} \qquad \underline{\underline{h_2(x) = -x+4}}$$

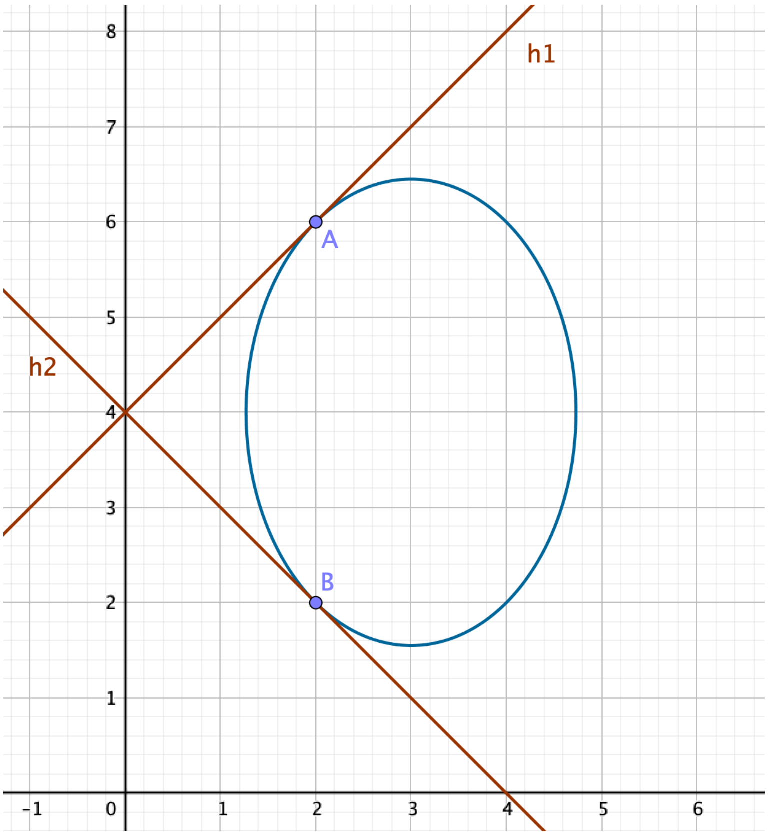

Figure 3: The ellipse with the two tangents for $x=2$