Solutions	EBA 29103 Mathematics for Business Analytics
Date	March 20th 2020 at 1200

Each sub-question has maximal score 6p, and the exam has maximal score 144p. To pass the exam, a score of approximately 86p is required. You may self-assess and use the table below to enter your scores. When we evaluate your answers, we emphasize the choice of method (you should give reasons based on theory where it is necessary), and the execution (that the computations are correct). To obtain the correct answer is less important, and there are in most cases alternative ways of writing the answer that will give full score.

Question	1.	2.	3.	4.	5.	6.	7.	8.	9.	Total	Grade	A	B	C	D	E
Points	24	12	12	18	18	18	24	6	12	144	Limits	130	110	84	66	58
Score																

Question 1. 24 P.

(a) $\int \frac{1}{\sqrt{x}}\,\mathrm{d}x = \int x^{-1/2}\,\mathrm{d}x = 2x^{1/2} + C = 2\sqrt{x} + C$ 6 P.

(b) $\int \frac{1-x}{x^2}\,\mathrm{d}x = \int \frac{1}{x^2} - \frac{x}{x^2}\,\mathrm{d}x = \int x^{-2} - x^{-1}\,\mathrm{d}x = -x^{-1} - \ln|x| + C = -1/x - \ln|x| + C$ 6 P.

(c) $\int \frac{x^2}{1-x}\,\mathrm{d}x = \int -x - 1 + \frac{1}{1-x}\,\mathrm{d}x = -\frac{1}{2}x^2 - x - \ln|1-x| + C$ 6 P.

(d) $\int 16(3-x)^7\,\mathrm{d}x = \int 16u^7 \cdot 1/(-1)\,\mathrm{d}u = -2u^8 + C = -2(3-x)^8 + C$ 6 P.

Question 2. 12 P.

(a) We write down the aumented matrix of the system, mark the first pivot position, and make elementary row operations using the first pivot to eliminate the entries below it:

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 6 \\ 2 & 0 & -1 & 1 \\ -1 & -2 & 6 & 7 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & -2 & 3 & 6 \\ 0 & 4 & -7 & -11 \\ -1 & -2 & 6 & 7 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & -2 & 3 & 6 \\ 0 & 4 & -7 & -11 \\ 0 & -4 & 9 & 13 \end{array}\right)$$

We mark the pivot position in the second row, and use it to eliminate the entries below it:

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 6 \\ 0 & 4 & -7 & -11 \\ 0 & -4 & 9 & 13 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & -2 & 3 & 6 \\ 0 & 4 & -7 & -11 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

We obtain an echelon form, where we have marked all pivot positions. Hence there is one solution, which we find by back substitution:

$$\begin{aligned} 2z &= 2 & & & z &= 1 \\ 4y - 7z &= -11 &\Rightarrow\quad 4y &= -11 + 7(1) & y &= -1 \\ x - 2y + 3z &= 6 &\Rightarrow\quad x &= 6 + 2(-1) - 3(1) & x &= 1 \end{aligned}$$

The solution is therefore $(x,y,z) = (1,-1,1)$. 6 P.

(b) We write down the aumented matrix of the system, mark the first pivot position, and make elementary row operations using the first pivot to eliminate the entries below it:

$$\left(\begin{array}{rrr|r} 1 & 2 & 4 & 5 \\ -3 & 1 & 5 & -5 \\ 5 & 3 & 3 & 15 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & 4 & 5 \\ 0 & 7 & 17 & 10 \\ 5 & 3 & 3 & 15 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & 4 & 5 \\ 0 & 7 & 17 & 10 \\ 0 & -7 & -17 & -10 \end{array}\right)$$

We mark the pivot position in the second row, and use it to eliminate the entries below it:

$$\left(\begin{array}{rrr|r} 1 & 2 & 4 & 5 \\ 0 & 7 & 17 & 10 \\ 0 & -7 & -17 & -10 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & 4 & 5 \\ 0 & 7 & 17 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We obtain an echelon form, where we have marked all pivot positions. Hence there are infinitely many solutions, with z free since there is no pivot in the z-column. We find them by back substitution:

$$\begin{aligned} 7y + 17z = 10 \quad &\Rightarrow \quad 7y = 10 - 17z & y &= 10/7 - 17z/7 \\ x + 2y + 4z = 5 \quad &\Rightarrow \quad x = 5 - 2(10/7 - 17z/7) - 4z & x &= 15/7 + 6z/7 \end{aligned}$$

The solutions are $(x,y,z) = (15/7 + 6z/7, 10/7 - 17z/7, z)$ where z is a free variable. 6 P.

Question 3. 12 P.

a) We compute the derivative of f and find that

$$\begin{aligned} f'(x) &= \frac{\frac{1}{2\sqrt{x}} \cdot (x+3) - (\sqrt{x}+1)\cdot 1}{(x+3)^2} = \frac{(x+3) - 2\sqrt{x}(\sqrt{x}+1)}{2\sqrt{x}(x+3)^2} \\ &= \frac{x + 3 - 2x - 2\sqrt{x}}{2\sqrt{x}(x+3)^2} = \frac{3 - 2\sqrt{x} - x}{2\sqrt{x}(x+3)^2} = \frac{(1-\sqrt{x})(\sqrt{x}+3)}{2\sqrt{x}(x+3)^2} \end{aligned}$$

To factorize the numerator, we put $u = \sqrt{x}$, which gives $3 - 2\sqrt{x} - x = 3 - 2u - u^2$ with the factorization $-(u-1)(u+3) = (1-u)(u+3) = (1-\sqrt{x})(\sqrt{x}+3)$. Since f is defined for $x \geq 0$, and all factors of $f'(x)$ except $1 - \sqrt{x}$ is positive for $x > 0$, we have that $f'(x) \geq 0$ i $(0,1]$ and $f'(x) \leq 0$ i $[1,\infty)$. This means that f is increasing in $[0,1]$ and decreasing in $[1,\infty)$. 6 P.

b) Siden f is increasing in $[0,1]$ and decreasing in $[1,\infty)$, it has a (global) maximum at $x = 1$, and the maximum value is $f_{\max} = f(1) = 2/4 = 1/2$. We have that $f(0) = 1/3$ and that $f(x) \to 0$ when $x \to \infty$. This means that f has no global minimum. 6 P.

Question 4. 18 P.

(a) We compute the determinant of A using cofactor expansion along the first row, and find that $|A| = 1(12+1) - 1(8-3) + 1(2+9) = 19$. This means that

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{19}\begin{pmatrix} 13 & -5 & 11 \\ -3 & 7 & -4 \\ -4 & 3 & 1 \end{pmatrix}^T = \frac{1}{19}\begin{pmatrix} 13 & -3 & -4 \\ -5 & 7 & 3 \\ 11 & -4 & 1 \end{pmatrix} \quad 6\text{ P.}$$

(b) We have that

$$A^T A = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 3 & 1 \\ 1 & -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ -3 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 14 & 4 & -13 \\ 4 & 11 & 2 \\ -13 & 2 & 18 \end{pmatrix} \quad 6\text{ P.}$$

(c) We have that

$$\begin{aligned} E_3 \cdot E_2 \cdot E_1 \cdot A &= E_3 \cdot E_2 \cdot \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ -3 & 1 & 4 \end{pmatrix} = E_3 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ -3 & 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 4 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 19 \end{pmatrix} \quad 6\text{ P.} \end{aligned}$$

Question 5. 18 P.

(a) Since $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x < 0$, we have that

$$\int_{-1}^{1} |x| \, dx = \int_{-1}^{0} (-x) \, dx + \int_{0}^{1} x \, dx = \left[-\frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{1}{2}x^2\right]_{0}^{1} = 0 + (1/2) + (1/2) - 0 = 1 \quad 6\text{ P.}$$

(b) We have that

$$\int_1^b \frac{1}{x^3}\,\mathrm{d}x = \int_1^b x^{-3}\,\mathrm{d}x = \left[-\frac{1}{2}x^{-2}\right]_1^b = -\frac{1}{2b^2} + \frac{1}{2}$$

This means that

$$\int_1^\infty \frac{1}{x^3}\,\mathrm{d}x = \lim_{b\to\infty}\left(\frac{1}{2} - \frac{1}{2b^2}\right) = \frac{1}{2} \quad 6\,\text{P.}$$

(c) Substitution gives $\int x\,e^{-x^2/2}\,\mathrm{d}x = \int e^{-u}\,\mathrm{d}u = -e^{-u} + C = -e^{-x^2/2} + C = F(x) + C$. Since $F(-b) = F(b)$, we have that

$$\int_{-\infty}^\infty x\,e^{-x^2/2}\,\mathrm{d}x = \lim_{b\to\infty}\int_{-b}^b xe^{-x^2/2}\,\mathrm{d}x = 0 \quad 6\,\text{P.}$$

Question 6. 18 P.

(a) The determinant is given by

$$|A| = \begin{vmatrix} 1 & 1 & a \\ 2 & a & 4 \\ 1 & 1 & 3 \end{vmatrix} = 1(3a-4) - 1(6-4) + a(2-a) = -a^2 + 5a - 6 = -(a-2)(a-3)$$

It follows that $|A| = 0$ when $a = 2$ or $a = 3$, and the linear system has exactly one solution when $a \neq 2, 3$. 6 P.

(b) The determinant is given by

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & a \\ a & 1 & a \end{vmatrix} = 1(a^2 - a) - 1(a - a^2) + 1(1 - a^2) = a^2 - 2a + 1 = (a-1)^2$$

Hence $|A| = 0$ when $a = 1$, and the linear system has exactly one solution when $a \neq 1$. 6 P.

(c) The determinant is given by

$$|A| = \begin{vmatrix} t & 1 & t \\ 1 & t & -1 \\ t & -1 & t \end{vmatrix} = t(t^2-1) - 1(t+t) + t(-1-t^2) = -4t$$

Hence $|A| = 0$ when $t = 0$, and the linear system has exactly one solution when $t \neq 0$. 6 P.

Question 7. 24 P.

(a) We use integration by parts with $u' = 5x\sqrt{x} = 5x^{3/2}$ and $v = \ln x$, which gives $u = 2x^{5/2} = 2x^2\sqrt{x}$ and $v' = 1/x$, and therefore

$$\int 5x\sqrt{x}\,\ln(x)\,\mathrm{d}x = 2x^2\sqrt{x}\,\ln(x) - \int 2x^{5/2}x^{-1}\,\mathrm{d}x = 2x^2\sqrt{x}\,\ln(x) - \int 2x^{3/2}\,\mathrm{d}x$$
$$= 2x^2\sqrt{x}\,\ln(x) - \frac{4}{5}x^2\sqrt{x} + C \quad 6\,\text{P.}$$

(b) We write the integral as $\int xe^{-x}\,\mathrm{d}x$ and use integration by parts with $u' = e^{-x}$ and $v = x$, which gives $u = -e^{-x}$ and $v' = 1$, and therefore

$$\int xe^{-x}\,\mathrm{d}x = x(-e^{-x}) - \int 1(-e^{-x})\,\mathrm{d}x = -xe^{-x} + \int e^{-x}\,\mathrm{d}x = -xe^{-x} - e^{-x} + C \quad 6\,\text{P.}$$

(c) We factor the denominator as $4 - x^2 = (2+x)(2-x)$, and simplify the expression using partial fractions. This gives

$$\frac{2x+2}{4-x^2} = \frac{A}{2+x} + \frac{B}{2-x} \quad \Rightarrow \quad 2x + 2 = A(2-x) + B(2+x)$$

Hence $2x + 2 = 2(A+B) + (B-A)x$, or $2(A+B) = 2$ and $B - A = 2$. These equations give $B = 3/2$ and $A = -1/2$, and the integral becomes

$$\int \frac{2x+2}{4-x^2}\,\mathrm{d}x = \int -\frac{1}{2}\frac{1}{2+x} + \frac{3}{2}\frac{1}{2-x}\,\mathrm{d}x = -\frac{1}{2}\ln|2+x| - \frac{3}{2}\ln|2-x| + C \quad 6\,\text{P.}$$

(d) We use the substitution $u = 1 - \sqrt{x}$, which gives $\mathrm{d}u = (-1/2)x^{-1/2}\,\mathrm{d}x$, or $\mathrm{d}x = -2\sqrt{x}\,\mathrm{d}u$. The integral becomes

$$\begin{aligned}\int \frac{\sqrt{x}}{1-\sqrt{x}}\,\mathrm{d}x &= \int \frac{\sqrt{x}}{u}(-2\sqrt{x})\,\mathrm{d}u = \int \frac{-2(\sqrt{x})^2}{u}\,\mathrm{d}u = \int \frac{-2(1-u)^2}{u}\,\mathrm{d}u \\ &= \int \frac{-2u^2+4u-2}{u}\,\mathrm{d}u = \int -2u+4-\frac{2}{u}\,\mathrm{d}u = -u^2+4u-2\ln|u|+C \\ &= -(1-\sqrt{x})^2+4(1-\sqrt{x})-2\ln|1-\sqrt{x}|+C = -x-2\sqrt{x}-2\ln|1-\sqrt{x}|+C\end{aligned}$$

In the first line, we have used that $u = 1-\sqrt{x}$, or $\sqrt{x} = 1-u$. 6 P.

Question 8. 6 P.

We first compute the determinant of the coefficient matrix A of the linear system by cofactor expansion along the first row:

$$|A| = \begin{vmatrix} t & 1 & 1 \\ 2 & 1 & t \\ 4 & t & 2 \end{vmatrix} = t(2-t^2) - 1(4-4t) + 1(2t-4) = 2t - t^3 - 4 + 4t + 2t - 4 = -t^3 + 8t - 8$$

We see that $t = 2$ gives $|A| = 0$, and we find the factorization

$$|A| = -(t^3 - 8t + 8) = -(t-2)(t^2+2t-4)$$

by polynomial division. Since $t^2 + 2t - 4 = 0$ has the solutions

$$t = \frac{-2 \pm \sqrt{4+16}}{2} = -1 \pm \sqrt{5}$$

we have that $\det(A) = 0$ for $t_1 = 2$, $t_2 = -1+\sqrt{5} \approx 1.23$ and $t_3 = -1-\sqrt{5} \approx -3.23$. This means that the system has no solutions or infinitely many solutions for these three values of t, and exactly one solution otherwise. We first consider the case $t = 2$, and solve the system by Gaussian elimination:

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 2 & 1 & 2 & 2 \\ 4 & 2 & 2 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We see that the system has infinitely many solutions for $t = 2$, with y as a free variable. The solutions are given by $z = 0$ and $2x + y + z = 2$, which gives $2x = 2 - y - z = 2 - y$, or $x = 1 - y/2$. For $t = 2$, the solutions are therefore

$$(x,y,z) = (1 - y/2, y, 0) \text{ where } y \text{ is a free variable}$$

We chech the two cases $t = -1 \pm \sqrt{5}$ simultaneously, using Gaussian eliminations. We first switch the rows and multiply the last row with 2:

$$\left(\begin{array}{ccc|c} t & 1 & 1 & 2 \\ 2 & 1 & t & t \\ 4 & t & 2 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 1 & t & t \\ 4 & t & 2 & 4 \\ t & 1 & 1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 1 & t & t \\ 4 & t & 2 & 4 \\ 2t & 2 & 2 & 4 \end{array}\right)$$

Then, we eliminate the entries under the first and second pivot:

$$\left(\begin{array}{ccc|c} 2 & 1 & t & t \\ 4 & t & 2 & 4 \\ 2t & 2 & 2 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 1 & t & t \\ 0 & t-2 & 2-2t & 4-2t \\ 0 & 2-t & 2-t^2 & 4-t^2 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 1 & t & t \\ 0 & t-2 & 2-2t & 4-2t \\ 0 & 0 & 4-2t-t^2 & 8-2t-t^2 \end{array}\right)$$

To find the pivots, we substitute $t = -1 \pm \sqrt{5}$. Then $t - 2 \neq 0$ is a pivot in the second row, and $4 - 2t - t^2 = 0$ because the two t-values are the solutions of $t^2 + 2t - 4 = 0$. But $8 - 2t - t^2 \neq 0$, hence the system has no solutions for $t = -1 \pm \sqrt{5}$. We may also see this by substituting approximate values for t_2 and t_3 and do the Gaussian elimination with the approximate values. For all values of t except

$t = 2$ and $t = -1 \pm \sqrt{5}$, there is exactly one solution, and we find it using Kramer's rule:

$$\begin{vmatrix} 2 & 1 & 1 \\ t & 1 & t \\ 4 & t & 2 \end{vmatrix} = -t^2 + 2t \qquad \Rightarrow \qquad x = \frac{-t(t-2)}{-(t-2)(t^2+2t-4)} = \frac{t}{t^2+2t-4}$$

$$\begin{vmatrix} t & 2 & 1 \\ 2 & t & t \\ 4 & 4 & 2 \end{vmatrix} = -2t^2 + 4t \qquad \Rightarrow \qquad y = \frac{-2t(t-2)}{-(t-2)(t^2+2t-4)} = \frac{2t}{t^2+2t-4}$$

$$\begin{vmatrix} t & 1 & 2 \\ 2 & 1 & t \\ 4 & t & 4 \end{vmatrix} = -t^3 + 12t - 16 \qquad \Rightarrow \qquad z = \frac{-(t-2)^2(t+4)}{-(t-2)(t^2+2t-4)} = \frac{(t-2)(t+4)}{t^2+2t-4}$$

To simplify the expression for z, we have used that $-t^3 + 12t - 16$ has one root $t = 2$, and polynomial division gives $-t^3 + 12t - 16 = (t-2)(-t^2 - 2t + 8) = -(t-2)^2(t+4)$. 6 P.

Question 9. 12 P.

a) We check whether $\mathbf{v}_4$ is a linear combination of the first three vectors by solving the vector equation $\mathbf{v}_4 = x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3$. This gives a linear system with the following extended matrix, and we start the Gaussian process:

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 2 & -1 & 0 & 12 \\ 1 & 2 & 4 & 5 \\ 3 & 1 & 3 & 16 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -7 & -4 & 10 \\ 0 & -1 & 2 & 4 \\ 0 & -8 & -3 & 13 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -1 & 2 & 4 \\ 0 & -7 & -4 & 10 \\ 0 & -8 & -3 & 13 \end{array}\right)$$

In the last row operation, we switch the second and third row. We complete the Gaussian process:

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -1 & 2 & 4 \\ 0 & -7 & -4 & 10 \\ 0 & -8 & -3 & 13 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & -18 & -18 \\ 0 & 0 & -19 & -19 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & -18 & -18 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We see that the linear system has a solution, and $\mathbf{v}_4$ is therefore a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. We find the solution of the linear system, which is given by

$$\begin{aligned} -18z &= -18 & & & z &= 1 \\ -y + 2z &= 4 & \Rightarrow \quad -y &= 4 - 2(1) & y &= -2 \\ x + 3y + 2z &= 1 & \Rightarrow \quad x &= 1 - 3(-2) - 2(1) & x &= 5 \end{aligned}$$

The solution is therefore $(x,y,z) = (5,-2,1)$, and this means that $\mathbf{v}_4 = 5\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3$. 6 P.

b) Since the vector $\mathbf{v}_4 = 5\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3$ is a linear combination of the other column vectors in A, it follows that $|A| = 0$. This means that

$$\det(A^T A) = \det(A^T)\det(A) = \det(A)\det(A) = 0 \quad 6\text{ P.}$$