Solutions: EBA 29103 2022/03

1. a) $\int_0^1 12x^2+3\sqrt{x}\,dx = \left[12\cdot\frac{1}{3}x^3 + 3\cdot\frac{2}{3}x^{3/2}\right]_0^1$

$= \left[4x^3+2x\sqrt{x}\right]_0^1 = (4+2)-0 = \underline{\underline{6}}$

b) $\int 9\sqrt{x}\ln x\,dx = \int 9x^{1/2}\ln x\,dx = 6x\sqrt{x}\ln x - \int 6x^{3/2}\cdot\frac{1}{x}\,dx$

$$\boxed{\begin{array}{ll} u = 9\cdot\frac{2}{3}x^{3/2} & v = \ln x \\ u' = 9x^{1/2} & v' = 1/x \end{array}}$$

$= 6x\sqrt{x}\ln x - 6\cdot\frac{2}{3}x^{3/2}+C = 6x\sqrt{x}\ln x - 4x\sqrt{x}+C$

$\Rightarrow \int_1^2 9\sqrt{x}\ln x\,dx = \left[6x\sqrt{x}\ln x - 4x\sqrt{x}\right]_1^2$

$=(12\sqrt{2}\ln 2 - 8\sqrt{2})-(0-4\cdot 1) = \underline{\underline{12\sqrt{2}\ln 2-8\sqrt{2}+4}}$

c) $\frac{6}{x^2-1} = \frac{A}{x-1}+\frac{B}{x+1} \Rightarrow 6 = A(x+1)+B(x-1) \Rightarrow A=3$

$= \underbrace{(A+B)}_{=0}x+\underbrace{(A-B)}_{=6} \qquad B=-3$

$\int \frac{6}{x^2-1}\,dx = 3\ln|x-1| - 3\ln|x+1| + C = 3\ln\left|\frac{x-1}{x+1}\right|+C$

$\Rightarrow \int_3^\infty \frac{6}{x^2-1}\,dx = \left[3\ln\left|\frac{x-1}{x+1}\right|\right]_3^\infty = 0 - (3\ln(2/4))$

$= -3\ln(1/2) = -3(-\ln 2) = \underline{\underline{3\ln 2}}$

since $\ln\left|\frac{x-1}{x+1}\right| \to \ln 1 = 0$ when $x\to\infty$.

d) $\int \frac{1}{\sqrt{x}+1}\,dx = \int \frac{1}{u}2\sqrt{x}\,du = \int \frac{1}{u}2(u-1)\,du = \int \frac{2u-2}{u}\,du$

$$\boxed{\begin{array}{l} u=\sqrt{x}+1 \\ du = \frac{1}{2\sqrt{x}}\,dx \end{array}}$$

$= \int 2-\frac{2}{u}\,du = 2u - 2\ln|u|+C$

$= 2(\sqrt{x}+1) - 2\ln(\sqrt{x}+1)+C$

$\Rightarrow \int_0^\infty \frac{1}{\sqrt{x}+1}\,dx = \left[2(\sqrt{x}+1) - 2\ln(\sqrt{x}+1)\right]_0^\infty = \lim_{b\to\infty}\left(2(\sqrt{b}+1)-2\ln(\sqrt{b}+1)\right)$

$-(2\cdot 1-2\ln 1) = \underline{\underline{\infty}}$

since $\sqrt{b}+1-\ln(\sqrt{b}+1)\to\infty$ when $b\to\infty$.

2.

a) $\dfrac{x+9}{x^2-3x-10} = \dfrac{A}{(x+2)} + \dfrac{B}{(x-5)} \Rightarrow x+9 = A(x-5) + B(x+2)$

$= \underset{\substack{\shortparallel \\ 1}}{(A+B)}x + \underset{\substack{\shortparallel \\ 9}}{(-5A+2B)}$

$A+B=1 \Rightarrow A = 1-B$

$-5A+2B=9$

$-5(1-B)+2B=9$

$\dfrac{7B}{7} = \dfrac{9+5}{7}$

$B=2$

$A=-1$

$\displaystyle\int \frac{x+9}{x^2-3x-10}\,dx = \int \frac{-1}{x+2} + \frac{2}{x-5}\,dx$

$= -\ln|x+2| + 2\ln|x-5| + C$

b) $\displaystyle\int \frac{\ln(x-1)}{x^2}\,dx = \int x^{-2}\ln(x-1)\,dx = -x^{-1}\ln(x-1) - \int -x^{-1}\frac{1}{x-1}\,dx$

$\boxed{\begin{array}{ll} u = -x^{-1} & v = \ln(x-1) \\ u' = x^{-2} & v' = \dfrac{1}{x-1} \end{array}}$

$= -\frac{1}{x}\ln(x-1) + \displaystyle\int \frac{1}{x(x-1)}\,dx$

$\dfrac{1}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1}$

$1 = A(x-1) + Bx$

$= \underset{\substack{\shortparallel \\ 0}}{(A+B)}x + \underset{\substack{\shortparallel \\ 1}}{(-A)}$

$A=-1,\ B=1$

$= -\frac{1}{x}\ln(x-1) + \displaystyle\int -\frac{1}{x} + \frac{1}{x-1}\,dx$

$= -\frac{1}{x}\ln(x-1) - \ln x + \ln(x-1) + C$

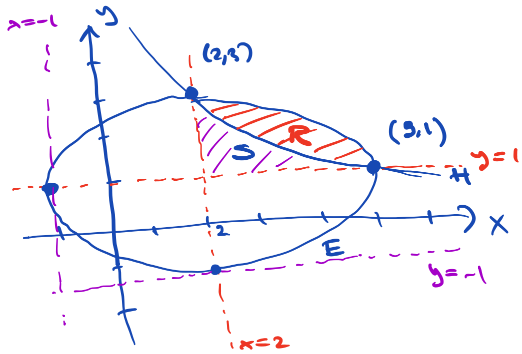
c) $\displaystyle\int \frac{1}{e^x+1}\,dx = \int \frac{1}{u}\cdot\frac{1}{e^x}\,du = \int \frac{1}{u}\cdot\frac{1}{(u-1)}\,du$

$\boxed{\begin{array}{l} u = e^x+1 \\ du = e^x\,dx \end{array}}$

From (b) $= \displaystyle\int -\frac{1}{u} + \frac{1}{u-1}\,du = -\ln|u| + \ln|u-1|$

$= -\ln(e^x+1) + \ln(e^x) + C = x - \ln(e^x+1) + C$

3.



a) $E: \quad \dfrac{(x-2)^2}{3^2} + \dfrac{(y-1)^2}{2^2} = 1 \quad$ or $\quad \underline{\underline{4(x-2)^2 + 9(y-1)^2 = 36}}$

$H: \quad (x+1)(y+1) = c \quad \Rightarrow (x-1)(y-1) = 12$

$(2,3): \quad (2+1)(3+1) = c$ or

$3 \cdot 4 = c \qquad y+1 = \dfrac{12}{x+1}$

$c = 12 \qquad \underline{\underline{y = -1 + \dfrac{12}{x+1}}}$

b) See figure above with region R marked.
We call the region bounded by $x=2, y=1$ and H for S,
such that $A(R) + A(S) = \dfrac{A(\text{ellipse})}{4} = \dfrac{\pi \cdot 3 \cdot 2}{4} = \dfrac{6}{4}\pi = \dfrac{3}{2}\pi$
we get

$$A(S) = \int_2^5 \left(-1 + \frac{12}{x+1}\right) - 1 \, dx = \int_2^5 \frac{12}{x+1} - 2 \, dx = \Big[12 \ln(x+1) - 2x\Big]_2^5$$

$$= (12 \ln 6 - 10) - (12 \ln 3 - 4) = 12 \ln\left(\frac{6}{3}\right) - 10 + 4 = \underline{12 \ln 2 - 6}$$

Hence the region R has area:

$$A(R) = \frac{3}{2}\pi - A(S) = \frac{3}{2}\pi - (12 \ln 2 - 6) = \underline{\underline{\frac{3}{2}\pi - 12 \ln 2 + 6}}$$

$$\simeq 2.39$$

4.

a) NPV: $\displaystyle\int_0^{10} f(x) e^{-rx} \, dx = \int_0^{10} (100+4x) e^{-0.10x} \, dx$

$$\boxed{\begin{array}{ll} u = -10e^{-0.1x} & v = 100+4x \\ u' = e^{-0.10x} & v' = 4 \end{array}}$$

$$= \Big[-10 e^{-0.1x} \cdot (100+4x)\Big]_0^{10} - \int_0^{10} -10 e^{-0.1x} \cdot 4 \, dx$$

$$= \Big[-10(100+4x) e^{-0.1x} + 40(-10) e^{-0.1x}\Big]_0^{10}$$

$$= \left(-1400 e^{-1} - 400 e^{-1}\right) - \left(-1000 e^{0} - 400 e^{0}\right) = 1400 - \frac{1800}{e} \simeq \underline{\underline{737.8}}$$

b) NPV: $\displaystyle\int_0^{10} f(x)e^{-rx}\,dx = \int_0^{10} 100\cdot 1.04^{x} e^{-0.10x}\,dx$

$$= \int_0^{10} 100\, e^{\ln(1.04)x - 0.10x}\,dx = \int_0^{10\ln(1.04)-1} 100\, e^{u}\cdot \frac{du}{\ln 1.04 - 0.1}$$

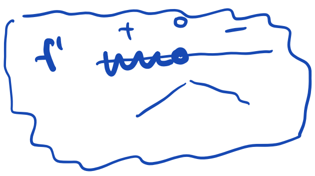
$$\boxed{\begin{array}{l} u = \ln(1.04)x - 0.10x \\ du = (\ln(1.04) - 0.10)\,dx \end{array}}$$

$$= \frac{100}{\ln 1.04 - 0.10}\Big[e^{u}\Big]_0^{10\ln(1.04)-1}$$

$$= \frac{100}{\ln 1.04 - 0.10}\left(e^{10\ln(1.04)-1} - 1\right)$$

$$= \underline{\underline{\frac{100}{\ln(1.04) - 0.10}\left(\frac{1.04^{10}}{e} - 1\right)}} \simeq 749.3$$

5. Local max where $f'(x) = 0$ and $f'(x)$ changes sign from + to ÷:



We see that this happens for

$\underline{x=0}$ and $\underline{x=3}$

We have $f(3) > f(0)$ since $\displaystyle f(3) - f(0) = \int_0^3 f'(x)\,dx$

$= -A_0 + A_1 = A_1 - A_0 > 0$

area over graph of $f'(x)$ in $[0,1]$

area under graph of $f'(x)$ on $[1,3]$

We see from the figure that $\underline{A_1 > A_0}$,

or $A_1 - A_0 > 0$. This means

that $\underline{\underline{x=3}}$ is the max. pt. for f.

Read off from the graph: $A_0 \simeq 2$ sq. $= 2\times(1/4)^2 = 2/16 = 1/8 = \underline{0.125}$

$A_1 \simeq 26$ sq. $= 26\,(1/4)^2 = 26/16 = 13/8 = \underline{1.625}$

Hence: $\displaystyle\int_0^3 f'(x)\,dx \simeq -0.125 + 1.625 = \underline{1.5}$

Note: We assume that f is defn. for x in $[-1.5, 3.25]$, the region shown in the fig. The boundary pt. $x=-1.5$ is then local max of f, but $\int_{-1.5}^{3} f'(x)\,dx > 0$ hence $f(-1.5) < f(3)$.

6.

a) $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 5 & 8 & -2 & 23 \\ 2 & 6 & -5 & 6 \\ 4 & 4 & 2 & 21 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & 8 \\ 0 & 2 & -3 & 0 \\ 0 & -4 & 6 & 9 \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & 8 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & -7 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & 8 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$7/8$

no solution

echelon form

b) $\left(\begin{array}{cccc|c} 1 & 2 & 4 & 1 & 11 \\ 2 & 5 & 4 & -3 & 18 \\ 4 & 8 & 12 & 0 & 28 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 4 & 1 & 11 \\ 0 & 1 & -4 & -5 & -4 \\ 0 & 0 & -4 & -4 & -16 \end{array}\right)$

echelon form

infinitely many sol's, one degree of freedom (w free)

$x + 2y + 4z + w = 11$
$y - 4z - 5w = -4$
$-4z - 4w = -16$

Back substitution: $\dfrac{-4z}{-4} = \dfrac{4w - 16}{-4}$

$z = -w + 4$

$y = -4 + 4z + 5w$
$= -4 + 4(-w+4) + 5w$
$= w + 12$

$x = 11 - 2y - 4z - w$
$= 11 - 2(w+12) - 4(-w+4) - w$
$= w - 29$

$\Rightarrow (x,y,z,w) = (w-29, w+12, -w+4, w)$ with w free

7. a)

$$\begin{vmatrix} 2 & 1 & 4 \\ 3 & 2 & 1 \\ 1 & a & 7 \end{vmatrix} = 2(14-a) - 1(21-1) + 4(3a-2)$$

$$= 28 - 2a - 20 + 12a - 8 = \underline{\underline{10a}}$$

$|A| = 0:$ $10a = 0$

$\underline{\underline{a = 0}}$

b)

$$\begin{vmatrix} 0 & s & 1 \\ s & 0 & 1 \\ 1 & 1 & s \end{vmatrix} = -s(s^2-1) + 1 \cdot s = -s^3 + 2s$$

$$= \underline{\underline{-s(s^2-2)}}$$

$|A| = 0:$ $-s\,(s^2-2) = 0$

$\underline{\underline{s = 0}}$ or $s = \underline{\underline{\pm\sqrt{2}}}$

c)

$$\left|\begin{array}{cc|cc} 1 & t & 0 & 0 \\ t & 1 & 0 & 0 \\ \hline 0 & 0 & t & 8 \\ 0 & 0 & 2 & t \end{array}\right| = \begin{vmatrix} 1 & t \\ t & 1 \end{vmatrix} \cdot \begin{vmatrix} t & 8 \\ 2 & t \end{vmatrix} = \underline{\underline{(1-t^2) \cdot (t^2-16)}}$$

$|A| = 0:$ $(1-t^2) \cdot (t^2-16) = 0$

$t = \underline{\underline{\pm 1}}$ or $t = \underline{\underline{\pm 4}}$

8. a) $\underline{w}$ linear comb. of $\underline{u_1}, \underline{u_2}, \underline{u_3}$ $\Longrightarrow$ $x\underline{u_1} + y\underline{u_2} + z\underline{u_3} = \underline{w}$ has solutions

Linear system:

$$\left(\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 3 & 5 & 11 & 4 \\ 2 & 6 & 4 & 6 \\ 4 & 7 & 9 & 2 \end{array}\right) \quad -3,\ -2,\ -4$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & -1 & -4 & -5 \\ 0 & 2 & -6 & 0 \\ 0 & -1 & -11 & -10 \end{array}\right) \quad 2,\ -1 \quad \to \left(\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & -1 & -4 & -5 \\ 0 & 0 & -14 & -10 \\ 0 & 0 & -7 & -5 \end{array}\right) \quad -1/2 \quad \to \left(\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & -1 & -4 & -5 \\ 0 & 0 & -14 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\underline{\underline{\text{Yes}}}$, $\underline{w}$ is a lin. comb. of $\underline{u_1}, \underline{u_2}, \underline{u_3}$ since the system has solutions.

b)

$$\left(\begin{array}{ccc|c} 1 & 2 & 5 & a \\ 3 & 5 & 11 & b \\ 2 & 6 & 4 & c \\ 4 & 7 & 9 & d \end{array}\right) \begin{array}{l} \cdot -3 \\ \cdot -2 \\ \cdot -4 \end{array} \to \left(\begin{array}{ccc|c} 1 & 2 & 5 & a \\ 0 & -1 & -4 & b-3a \\ 0 & 2 & -6 & c-2a \\ 0 & -1 & -11 & d-4a \end{array}\right) \begin{array}{l} 2 \\ -1 \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 5 & a \\ 0 & -1 & -4 & b-3a \\ 0 & 0 & -14 & c-2a+2(b-3a) \\ 0 & 0 & -7 & d-4a-1(b-3a) \end{array}\right) \cdot 2$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 5 & a \\ 0 & -1 & -4 & b-3a \\ 0 & 0 & -14 & c+2b-8a \\ 0 & 0 & -14 & 2d-2b-2a \end{array}\right) -1$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 5 & a \\ 0 & -1 & -4 & b-3a \\ 0 & 0 & -14 & c+2b-8a \\ 0 & 0 & 0 & 2d-2b-2a-1\cdot(c+2b-8a) \end{array}\right)$$

There is a solution $\iff$ $2d - c - 4b + 6a = 0$ (no solutions otherwise)

$$\underline{c = 6a - 4b + 2d}$$

All linear comb. of $\underline{v}_1, \underline{v}_2, \underline{v}_3$: all vectors (a, b, c, d) such that

$$\underline{\underline{c = 6a - 4b + 2d}}$$

9. a) $a=0$

$$A = \begin{pmatrix} 3 & 7 & 0 \\ 2 & 5 & 3 \\ 5 & 0 & 35 \end{pmatrix}$$

$$|A| = 3(5\cdot 35) - 7\cdot(70-15) = 3\cdot 175 - 7\cdot 55 = 525 - 385 = \underline{140} \neq 0$$

$$A^{-1} = \frac{1}{140}\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{140}\begin{pmatrix} 175 & -55 & -25 \\ -245 & 105 & 35 \\ 21 & -9 & 1 \end{pmatrix}^T$$

$$= \frac{1}{140}\begin{pmatrix} 175 & -245 & 21 \\ -55 & 105 & -9 \\ -25 & 35 & 1 \end{pmatrix}$$

b) $|A| \neq 0 \iff A\underline{x} = \underline{b}$ has a unique solution

$$\begin{vmatrix} 3 & 7 & a \\ 2 & 5 & 3 \\ 5 & a & 35 \end{vmatrix} = 3\cdot(175-3a) - 7(70-15) + a(2a-25)$$

$$= 2a^2 - 34a + 525 - 385$$

$$= 2(a^2 - 17a + 70) = \underline{2(a-7)(a-10)}$$

Unique solution for $\underline{a \neq 7, 10}$

c) $\underline{a=7}$:

$$\left(\begin{array}{ccc|c} 3 & 7 & 7 & -8 \\ 2 & 5 & 3 & 4 \\ 5 & 7 & 35 & -144 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & -12 \\ 2 & 5 & 3 & 4 \\ 5 & 7 & 35 & -144 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & -12 \\ 0 & 1 & -5 & 28 \\ 0 & -3 & 15 & -84 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & -12 \\ 0 & 1 & -5 & 28 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Infinitely many solutions for $\underline{a=7}$

$$x + 2y + 4z = -12$$
$$y - 5z = 28$$

$$x = -12 - 4z - 2(5z+28) = \underline{-14z - 68}$$
$$y = \underline{5z + 28}$$

$(x,y,z) = \underline{(-14z-68, 5z+28, z)}$ with z free

$\underline{a=10:}$

$$\left(\begin{array}{ccc|c} 3 & 7 & 10 & -8 \\ 2 & 5 & 3 & 4 \\ 5 & 10 & 35 & -144 \end{array}\right) \begin{matrix}]-1 \\ \\ \\ \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 7 & -12 \\ 2 & 5 & 3 & 4 \\ 5 & 10 & 35 & -144 \end{array}\right) \begin{matrix}]-2 \\ \\ ]-5 \end{matrix}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 7 & -12 \\ 0 & 1 & -11 & 28 \\ 0 & 0 & 0 & -84 \end{array}\right)$$

<u>no solutions</u> for <u>$a=10$</u>

d) Unique solution: $a \neq 7, 10$

$$z = \frac{\begin{vmatrix} 3 & 7 & -8 \\ 2 & 5 & 4 \\ 5 & a & -144 \end{vmatrix}}{|A|} = \frac{-28a+196}{2(a-7)(a-10)} = \frac{-28\cancel{(a-7)}}{2\cancel{(a-7)}(a-10)} = \underline{\underline{\frac{-14}{a-10}}}$$

$$\begin{vmatrix} 3 & 7 & -8 \\ 2 & 5 & 4 \\ 5 & a & -144 \end{vmatrix} = 3(-720-4a) - 7(-288-20) - 8(2a-25)$$

$$= \underline{-28a+196}$$

10. We write: $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$D = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$

$|A| = -1-1 = -2 \neq 0$

$A^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}A$

<u>Matrix equation:</u>

$AXA = D \quad | A^{-1}\cdot$

$A^{-1}AXA = A^{-1}D$

$XA = A^{-1}D \quad | \cdot A^{-1}$

$XAA^{-1} = A^{-1}DA^{-1}$

$X = A^{-1}DA^{-1} = \frac{1}{2}A \cdot D \cdot \frac{1}{2}A = \frac{1}{4}ADA$

$= \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix} 6 & 2 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}}$