Solutions	EBA 2910 Mathematics for Business Analytics
Date	June 19th 2020 at 0900 - 1600

Question 1.

a) The slope of the tangent of f at $x = 1$ is given by $f'(1)$. We read off the graph of f' that $f'(1) = 0$. In a similar way, we read off the graph of g' that $g'(1) = 2$.

b) We see from the graph of f' that $f'(x) < 0$ when $x < 1$ and that $f'(x) > 0$ when $x > 1$. This means that f is decreasing for $x \leq 1$ and increasing for $x \geq 1$. Hence, $x = 1$ is a minimum point for f. In a similar way, we see from the graph of g' that $g'(x) \geq 0$ for all x, and g is therefore an increasing function. It follows that $x = -1$ is a minimum point for g.

c) In b) we found when f and g are increasing and decreasing. We conclude that g has an inverse function since g is increasing in the domain of definition, while f does not have an inverse function since f changes from being decreasing to being increasing at $x = 1$.

Question 2.

a) We write down the extended matrix of the system for $a = 2$, mark the first pivot and add the last row to the first to get the first pivot to be equal to 1 (which makes the computations easier). Then we use the first pivot to get zeros in the two positions under it:

$$\left(\begin{array}{rrr|r} 3 & 2 & -2 & 1 \\ 2 & 5 & -2 & 0 \\ -2 & -2 & 3 & 1 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & 1 & 2 \\ 2 & 5 & -2 & 0 \\ -2 & -2 & 3 & 1 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & 1 & 2 \\ 0 & 5 & -4 & -4 \\ 0 & -2 & 5 & 5 \end{array}\right)$$

We mark the second pivot, in the second row, and add 2 times the last row to the second row to get the second pivot to be equal to 1. Then we use the second pivot to eliminate the number in the position below it:

$$\left(\begin{array}{rrr|r} 1 & 0 & 1 & 2 \\ 0 & 5 & -4 & -4 \\ 0 & -2 & 5 & 5 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & 1 & 2 \\ 0 & 1 & 6 & 6 \\ 0 & -2 & 5 & 5 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & 1 & 2 \\ 0 & 1 & 6 & 6 \\ 0 & 0 & 17 & 17 \end{array}\right)$$

The result is an echelon form, where we have marked all pivots. We see that there is exactly one solutions, and we find it using back substitution:

$$\begin{aligned} 17z &= 17 & & & z &= 1 \\ y + 6z &= 6 &\Rightarrow\quad & y = 6 - 6(1) & y &= 0 \\ x + z &= 2 &\Rightarrow\quad & x = 2 - (1) & x &= 1 \end{aligned}$$

The solutions is therefore $(x,y,z) = (1,0,1)$.

b) We compute the determinant using cofactor expansion along the first row:

$$|A| = \begin{vmatrix} 3 & a & -2 \\ a & a^2+1 & -a \\ -2 & -a & 3 \end{vmatrix} = 3(3(a^2+1) - a^2) - a(3a - 2a) + (-2)(-a^2 + 2(a^2+1))$$
$$= 6a^2 + 9 - a^2 - 2a^2 - 4 = 3a^2 + 5$$

The system has exactly one solution when $\det(A) \neq 0$, and this happens for all values of a since $\det(A) = 3a^2 + 5 = 0$ does not have any solutions.

c) When $a = 0$, we have that $|A| = 3(0) + 5 = 5$. The inverse matrix is therefore given by

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{5}\begin{pmatrix} 3 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 3 \end{pmatrix}^T = \frac{1}{5}\begin{pmatrix} 3 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

d) From a) we have that $A\mathbf{x} = \mathbf{b}$ has solution $\mathbf{x} = \mathbf{b}$ when $a = 2$. Therefore $A\mathbf{b} = \mathbf{b}$, and by repeating this we obtain $A^2\mathbf{b} = A(A\mathbf{b}) = A\mathbf{b} = \mathbf{b}$, $A^3\mathbf{b} = A(A^2\mathbf{b}) = A\mathbf{b} = \mathbf{b}$, and this pattern continues. Therefore, we have that $A^n\mathbf{b} = \mathbf{b}$ for all $n \geq 1$.

Question 3.

a) We see from the graph of f that $f(x) \to \pm\infty$ when $x \to 2$. This means that $x = 2$ is a vertical asymptote for f. We also see that there is a straight line $y = L(x)$ such that $f(x) - L(x) \to 0$ when $x \to \pm\infty$ (which means that the graph of f will approach this line when $x \to \pm\infty$), and we read off from the graph that this straight line is given by $L(x) = -x - 2$ since it intersects the y-axis at $y = -2$ and it has slope -1. It follows that $y = -x - 2$ is a skrew asymptote for f.

b) The functional expression of f has quadratic numerator and linear denominator, og from the asymptotes of f we know that it has the form

$$f(x) = -x - 2 + \frac{C}{x-2}$$

for a constant C. We read off the point $(x,y) = (1,-2)$ from the graph of f, and use it to find C. This gives the equation $f(1) = -1 - 2 + C/(1-2) = -3 - C = -2$, which means that $C = -1$. Hence we have that

$$f(x) = -x - 2 + \frac{-1}{x-2} = \frac{(-x-2)(x-2)-1}{x-2} = \frac{3-x^2}{x-2}$$

We find $f'(x)$ starting from the first expression, and using the chain rule we get

$$f'(x) = -1 - 1(-1)(x-2)^{-2} \cdot 1 = -1 + \frac{1}{(x-2)^2}$$

c) Possible extremal points must be stationary points of f. We find two stationary points:

$$f'(x) = -1 + \frac{1}{(x-2)^2} = 0 \quad \Rightarrow \quad (x-2)^2 = 1 \quad \Rightarrow \quad x = 2 \pm 1$$

We find these points on the graph of f, and see that $x = 3$ is a local maximum but not a global maximum, and that $x = 1$ is a local minimum but not a global minimum. Hence f has no global maxima or minima.

Question 4.

a) We expand the polynomial and use the power rule:

$$\int x(1-x)^2 \, \mathrm{d}x = \int x(1-2x+x^2) \, \mathrm{d}x = \int x - 2x^2 + x^3 \, \mathrm{d}x = \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 + C$$

b) We use the substitution $u = 1 - x^2$, which gives $\mathrm{d}u = -2x\,\mathrm{d}x$, and the integral becomes

$$\int \frac{x}{1-x^2} \, \mathrm{d}x = \int \frac{x}{u} \cdot \frac{\mathrm{d}u}{-2x} = \int \frac{1}{-2u} \, \mathrm{d}u = -\frac{1}{2}\ln|u| + C = -\frac{1}{2}\ln|1-x^2| + C$$

c) We use the substitution $u = 1 - \sqrt{x}$, which gives $\mathrm{d}u = -\,\mathrm{d}x/(2\sqrt{x})$, and the integral becomes

$$\int \frac{x}{(1-\sqrt{x})^2} \, \mathrm{d}x = \int \frac{x}{u^2} \cdot (-2\sqrt{x}) \, \mathrm{d}u = \int \frac{-2(\sqrt{x})^3}{u^2} \, \mathrm{d}u = -2 \int \frac{(1-u)^3}{u^2} \, \mathrm{d}u$$

since $u = 1 - \sqrt{x}$ gives $\sqrt{x} = 1 - u$. We expand $(1-u)^3 = 1 - 3u + 3u^2 - u^3$, and integrate term by term using the power rule, and get

$$\int \frac{(1-3u+3u^2-u^3)}{u^2} \, \mathrm{d}u = \int \frac{1}{u^2} - \frac{3}{u} + 3 - u \, \mathrm{d}u = -\frac{1}{u} - 3\ln|u| + 3u - \frac{u^2}{2} + C$$

We substitute back $u = 1 - \sqrt{x}$, and this gives

$$\int \frac{x}{(1-\sqrt{x})^2} \, \mathrm{d}x = \frac{2}{1-\sqrt{x}} + 6\ln|1-\sqrt{x}| - 6(1-\sqrt{x}) + (1-\sqrt{x})^2 + C$$

d) We want to maximize the function $A(a) = \int_{-4}^{a} f(x)\,\mathrm{d}x$ with $-4 \le a \le 3$. We know from the theory of integration that $A'(a) = f(a)$, and the stationary points of A are therefore the zeros of f, which means $a = -3$, $a = -1$ og $a = 2$. Since $A'(a) = f(a)$ changes sign from negative to positive at $a = -1$, this is a local minimum. Similarly, $A'(a) = f(a)$ changes sign from positive to negative at $a = -3$ and $a = 2$, hence these points are local maxima for A. To compare $A(-3)$ and $A(2)$, we look at the integral

$$\int_{-3}^{2} f(x)\,\mathrm{d}x = \int_{-3}^{-1} f(x)\,\mathrm{d}x + \int_{-1}^{-2} f(x)\,\mathrm{d}x$$

It has a positive value since the area over the graph in $[-3, -1]$ is smaller than the area under the graph in $[-1,2]$. Therefore $A(2) > A(-3)$, and we see that the endpoints $a = -4$ and $a = 3$ does not give a maximum. We conclude that $a = 2$ is the maximum point for A, and therefore $a = 2$ gives the maximum value of the integral.

Question 5.

a) We get $f'(x) = 1 \cdot e^x + x \cdot e^x = (x + 1)e^x$. A sign diagram for $f'(x)$ shows that f is decreasing for $x \le -1$ and increasing for $x \ge -1$, hence $f_{\min} = f(-1) = -e^{-1} = -1/e \approx -0.37$ is the global minimum value of f. Since $-1 < f_{\min} \approx -0.37$, the equation $f(x) = -1$ has no solution.

b) Since $W = f^{-1}$, we have that $W(f(x)) = x$. The chain rule gives $W'(f(x)) \cdot f'(x) = 1$. From a) we have $f'(x) > 0$ for $x > -1$, hence $W'(f(x)) > 0$ for $x > -1$, and W is an increasing function.

Question 6.

a) We complete the square to rewrite and simplify the equation $4x^2 - 24x + t^2y^2 = 64$ for C:

$$4(x^2 - 6x + 9) + t^2y^2 = 64 + 4 \cdot 9$$
$$4(x-3)^2 + t^2y^2 = 100$$
$$\frac{(x-3)^2}{25} + \frac{t^2y^2}{100} = 1$$

If $t \ne 0$, then C is an ellipse with center (3,0) and half-axis $a = 5$ and $b = \sqrt{100/t^2} = 10/|t|$. For $t = \pm 2$, C is in particular a circle. If $t = 0$, then C consists of the two straight lines $x = -2$ and $x = 8$:

$$(x-3)^2 = 25 \quad \Rightarrow \quad x - 3 = \pm 5 \quad \Rightarrow \quad x = 3 \pm 5$$

The curve C for $t = 0$ (gray), $t = \pm 1$ (red), $t = \pm 2$ (blue), $t = \pm 4$ (green) is shown in Figure 1.

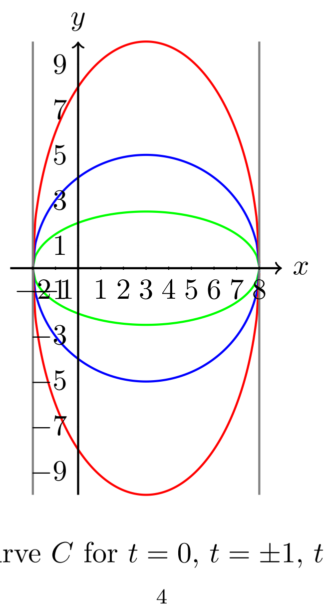

FIGURE 1. The curve C for $t = 0$, $t = \pm 1$, $t = \pm 2$, and $t = \pm 4$

b) We find the partial derivatives of $f(x,y) = xy$, given by $f'_x = y$ and $f'_y = x$. The first order conditions are $f'_x = f'_y = 0$, and this gives one stationary point $(x,y) = (0,0)$. The second order partial derivatives and the Hessian matrix is given by

$$H(f) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For $(x,y) = (0,0)$, we have $\det H(f) = -1 < 0$. The second derivative test gives that (0,0) is a saddle point.

c) The constraint $4x^2 - 24x + 16y^2 = 64$ gives the curve C for $t = \pm 4$. Since this is an ellipse, it is bounded, and the problem has a maximum by the Extreme Value Theorem. We also see that there are no admissible points with degenerate constraint since the constraint gives an ellipse. Therefore, the maximum point must be an ordinary candidate point. We find these using Lagrange's method: The Lagrangian is $\mathcal{L} = xy - \lambda(4x^2 - 24x + 16y^2)$, and the Lagrange conditions are the first order conditions (FOC) and the constraint (C), given by

$$\mathcal{L}'_x = y - \lambda(8x - 24) = 0$$
$$\mathcal{L}'_y = x - \lambda(32y) = 0$$
$$4x^2 - 24x + 16y^2 = 64$$

We solve the second equation for λ, and substitute this into the first equation. We get

$$\lambda = \frac{x}{32y} \quad \Rightarrow \quad y = \frac{x}{32y}(8x - 24)$$

Before continuing, we check if there are any solutions with $y = 0$: If $y = 0$, then $x = \lambda(32y) = 0$, and $(x,y) = (0,0)$ does not satisfy the constraint. Hence, there are no solutions with $y = 0$, and we can divide by y. In the equation displayed above, we multiply with the common denominator, and this gives $32y^2 = x(8x - 24) = 8x(x - 3)$, or $16y^2 = 4x(x - 3)$. Substituted into the constraint, this gives $4x^2 - 24x + 4x(x - 3) = 64$, which can be written $8x^2 - 36x - 64 = 0$ or $2x^2 - 9x - 16 = 0$. Hence, we find the solutions

$$x = \frac{9 \pm \sqrt{81 + 128}}{4} = \frac{9 \pm \sqrt{209}}{4}$$

With approximate values, this gives $x_1 \approx 5.864$ og $x_2 \approx -1.364$. For each solution for x, we find two solutions for y by using that $16y^2 = 4x(x - 3)$. We get

$$y^2 = \frac{x^2 - 3x}{4} \quad \Rightarrow \quad y = \pm\sqrt{x^2 - 3x}/2$$

When $x = x_1$, this gives $y \approx \pm 2.049$, and when $x = x_2$, this gives $y \approx \pm 1.220$. There are therefore four candidate points:

$$(x,y) \approx (5.864, \pm 2.049), (-1.364, \pm 1.220)$$

Among these points, $(x,y) \approx (5.864, 2.049)$ gives the greatest value of $f(x,y) = xy$. We may therefore conclude that the maximum value is

$$f_{\max} \approx f(5.864, 2.049) = 5.864 \cdot 2.049 \approx 12.02$$

d) The level curves for $f(x,y) = xy$ has the form $xy = c$, or $y = c/x$ for $c \neq 0$, and these curves are hyperbolas. For $c = 0$, the level curve consists of the two axis. The level curves for $c = \pm 16$, $c \approx \pm 12.0$ (the maximum value), $c = \pm 8$, $c = \pm 4$ og $c \approx \pm 1.7$ (the value at the other candidate points) are shown as blue hyperbolas in Figure 2, together with the constraint (red ellipse) and the candidate points. The function value of each level curve is shown (gray). We see that at each candidate point, the level curve of f meets the ellipse at a tangent, and only these points may solve the Lagrange problem: Level curves that don't meet the ellipse (dashed) have no admissible points. Level curves that meet the ellipse, but not at a tangent, cannot gives maximum points since the level curve will still meet the ellipse if we increase c slightly.

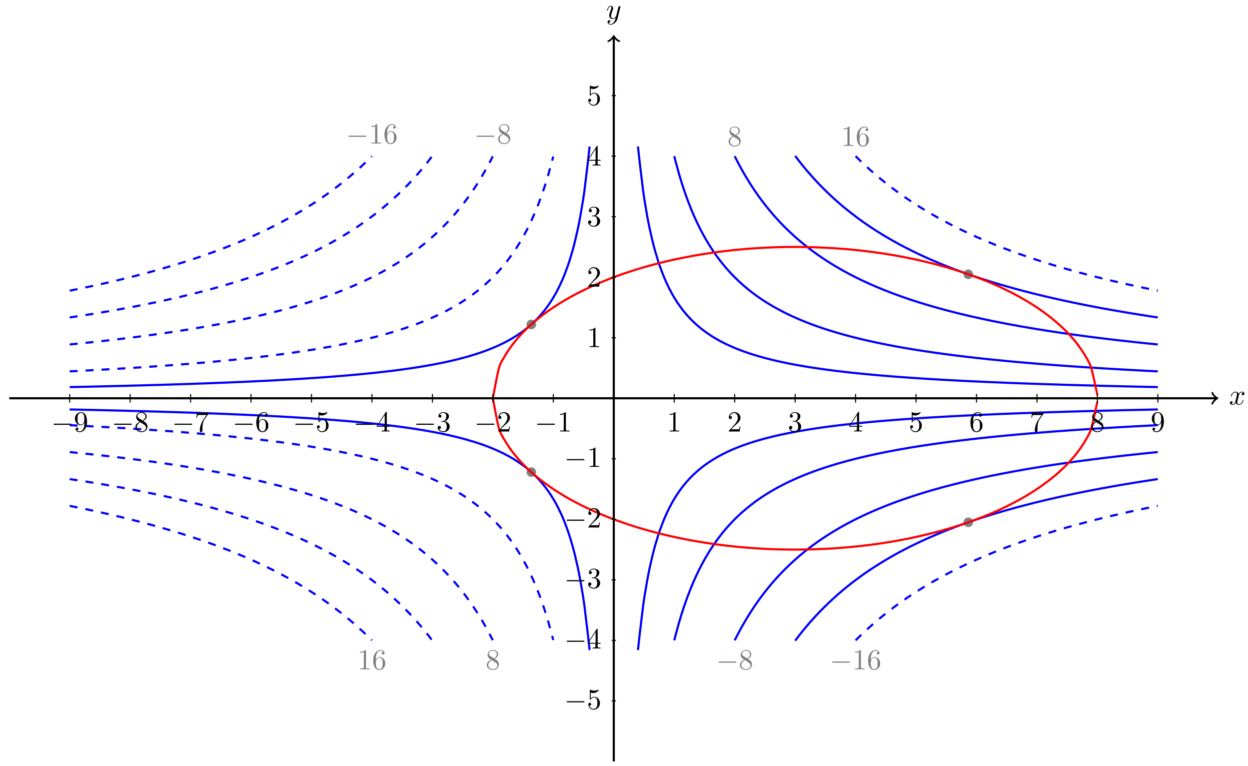

FIGURE 2. The level curves $xy = c$ for $c = \pm 16, \approx \pm 12.0, \pm 8, \pm 4, \approx \pm 1.7$