EVALUATION GUIDELINES - Take-home examination

EBA 09101
Mathematics for Business Analytics

Department of Economics

Start date:	02.06.2021	Time 10:00
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For more information about formalities, see examination paper.

Solutions	EBA 2910 Mathematics for Business Analytics
Date	June 2nd 2021 at 1000 - 1500

Question 1.

We write the functional expression $f(x) = 2\sqrt{x}\ln(x) - 4\sqrt{x} = 2(\ln x - 2)\sqrt{x}$, and see that $f(x)$ is defined for $x > 0$.

(a) We use the product rule to differentiate $f(x) = 2(\ln x - 2)\sqrt{x}$:

$$f'(x) = \frac{2}{x}\sqrt{x} + 2(\ln x - 2)\cdot\frac{1}{2\sqrt{x}} = \frac{2}{\sqrt{x}} + \frac{\ln x - 2}{\sqrt{x}} = \frac{2 + \ln x - 2}{\sqrt{x}} = \frac{\ln x}{\sqrt{x}}$$

(b) When $x \to \infty$, we have that $2(\ln x - 2) \to \infty$ and $\sqrt{x} \to \infty$. This gives

$$\lim_{x\to\infty} f(x) = \infty$$

When $x \to 0^+$, we have that $2(\ln x - 2) \to -\infty$ and $\sqrt{x} \to 0$. We therefore write the functional expression $f(x) = 2(\ln x - 2)\sqrt{x}$ as a fraction and use L'Hôspital's rule to compute the limit:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{2(\ln x - 2)}{1/\sqrt{x}} = \lim_{x\to 0^+} \frac{2/x}{(-1/2)x^{-3/2}} = \lim_{x\to 0^+} \frac{-4\sqrt{x}}{1} = 0$$

(c) We have that $f'(x) = 0$ when $\ln x = 0$, which gives $x = 1$, and from the expression for $f'(x)$ found in (a), we see that f is decreasing in (0,1] and increasing in [1,∞). Hence f has a global minimum $f(1) = -4$, and from the results in (b), f will tend to 0 when $x \to 0^+$ and tend to ∞ when $x \to \infty$. We conclude that $V_f = [-4, \infty)$, and that $f(x) = a$ has two solutions for $-4 < a < 0$. This gives

$$\text{The equation } f(x) = a \text{ has } \begin{cases} \text{two solutions for } -4 < a < 0 \\ \text{one solution for } a = -4 \text{ and } a \geq 0 \\ \text{no solutions for } a < -4 \end{cases}$$

The graph of f is shown below. As justification for the answer, one may use a sketch of the graph drawn by hand, but the arguments above (or similar) must be part of the justification.

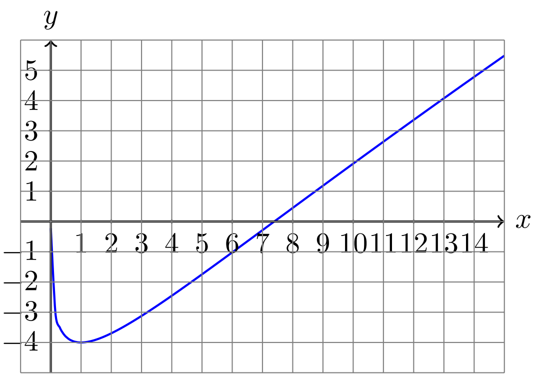


FIGURE 1. The graph of $y = f(x)$

Question 2.

(a) We use the factorization $9 - x^2 = (3 - x)(3 + x)$ and the partial fractions decomposition to write the expression under the integral as

$$\frac{3 - 7x}{9 - x^2} = \frac{A}{3 - x} + \frac{B}{3 + x}$$

This gives $3 - 7x = A(3 + x) + B(3 - x) = (3A + 3B) + (A - B)x$, hence we have that $3A + 3B = 3$, or $A + B = 1$, and that $A - B = -7$. Adding these equations, we get $2A = -6$, or $A = -3$, and this gives $B = 4$. Therefore, the integral takes the form

$$\int \frac{3 - 7x}{9 - x^2}\,\mathrm{d}x = \int \frac{-3}{3 - x} + \frac{4}{3 + x}\,\mathrm{d}x = 3\ln|3 - x| + 4\ln|3 + x| + C$$

(b) We use the substitution $u = x + 1$, which gives $\mathrm{d}u = \mathrm{d}x$, and the integral takes the form

$$\int 15x \cdot \sqrt{x+1}\,\mathrm{d}x = \int 15(u-1)\sqrt{u}\,\mathrm{d}u = \int 15u\sqrt{u} - 15\sqrt{u}\,\mathrm{d}u = \int 15u^{3/2} - 15u^{1/2}\,\mathrm{d}u$$
$$= 15 \cdot \frac{2}{5}u^{5/2} - 15 \cdot \frac{2}{3}u^{3/2} + C = 6u^2\sqrt{u} - 10u\sqrt{u} + C$$
$$= \sqrt{x+1} \cdot (6(x+1)^2 - 10(x+1)) + C = (6x^2 + 2x - 4)\sqrt{x+1} + C$$

(c) We use the substitution $u = \ln(x)$, which gives $\mathrm{d}u = (1/x)\,\mathrm{d}x$, or $\mathrm{d}x = x\,\mathrm{d}u$, and the integral takes the form

$$\int \frac{3\sqrt{\ln x}}{x}\,\mathrm{d}x = \int \frac{3\sqrt{u}}{x} \cdot x\,\mathrm{d}u = \int 3u^{1/2}\,\mathrm{d}u = 3 \cdot \frac{2}{3}u^{3/2} + C = 2\ln x\,\sqrt{\ln x} + C$$

Question 3.

(a) Since the graph of $f'(x)$ is a hyperbola, the functional expression of $f'(x)$ can be written in the form

$$f'(x) = c + \frac{a}{x-b}$$

with a vertical asymptote $x = b$ and a horisontal asymptote $y = c$. We read off from the graph as precisely as we can, and find that the vertical asymptote is $x = 1.25 = 5/4$ and that the horisontal asymptoten is $y = 1$. A natural degree of precision is the nearest multiple of $0.25 = 1/4$ given the grid shown in the figure. This gives

$$f'(x) = 1 + \frac{a}{x - 5/4} = 1 + \frac{4a}{4x-5}$$

We read off a point on the graph to determine a, and choose the intersection with the y-axis. We read this point off as $(1.75, 0) = (7/4, 0)$. This gives

$$0 = 1 + \frac{4a}{4 \cdot 7/4 - 5} = 1 + \frac{4a}{2} \quad \Rightarrow \quad 4a = -2$$

We can also write this as $a = -1/2$. This gives the functional expression

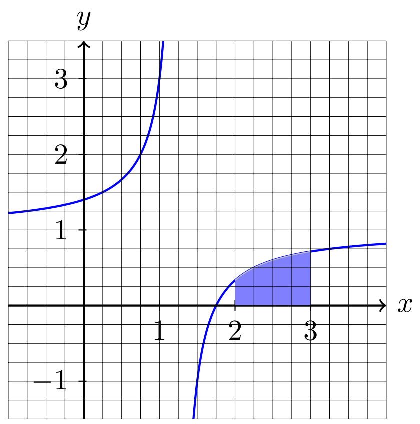
$$f'(x) = 1 + \frac{-2}{4x-5} = \frac{4x-5-2}{4x-5} = \frac{4x-7}{4x-5}$$

(b) By definition, we have that $f(x)$ is an antiderivative of the function $f'(x)$, and this means that

$$\int_2^3 f'(x)\,\mathrm{d}x = [f(x) + C]_2^3 = f(3) - f(2)$$

Hence $f(3) - f(2)$ is the area under the graph of $y = f'(x)$ in the interval [2,3], marked in the figure below. We read off an approximate value of $f(3) - f(2)$ by counting the number of squares under the graph. Each square has area $1/4 \cdot 1/4 = 1/16$, and since the area under the graph is approximately equal to 9 squares, we find the estimate

$$f(3) - f(2) \approx 9 \cdot \frac{1}{16} = \frac{9}{16} \approx 0.56$$



(c) We find an functional expression for $f(x)$ using the expression of $f'(x)$ from (a) and that $f(x)$ is an antiderivative for $f'(x)$. This gives

$$f(x) = \int f'(x)\,\mathrm{d}x = \int 1 + \frac{-2}{4x-5}\,\mathrm{d}x = x - 2\ln|4x-5| \cdot \frac{1}{4} + C = x - \frac{1}{2}\ln|4x-5| + C$$

Based on this expression, we find the value

$$f(3) - f(2) = \left[x - \frac{1}{2}\ln|4x-5|\right]_2^3 = (3 - \ln(7)/2) - (2 - \ln(3)/2)$$
$$= 1 - \frac{1}{2}(\ln 7 - \ln 3) = 1 - \frac{1}{2}\ln(7/3) \approx 0.576$$

Question 4.

(a) We compute the determinant using cofactor expansion along the first row:

$$|A| = \begin{vmatrix} 3 & 4 & 5 \\ 7 & 2 & 11 \\ 5 & 1 & 6 \end{vmatrix} = 3(12-11) - 4(42-55) + 5(7-10) = 3 + 52 - 15 = 40$$

This means that A has an inverse, and we use the adjungated matrix to find it:

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{40}\begin{pmatrix} 1 & 13 & -3 \\ -19 & -7 & 17 \\ 34 & 2 & -22 \end{pmatrix}^T = \frac{1}{40}\begin{pmatrix} 1 & -19 & 34 \\ 13 & -7 & 2 \\ -3 & 17 & -22 \end{pmatrix}$$

(b) Since A is invertible, the system $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} = A^{-1} \cdot \mathbf{b}$. When $r = 24$, $s = -20$, and $t = -6$, the solution is given by

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{40}\begin{pmatrix} 1 & -19 & 34 \\ 13 & -7 & 2 \\ -3 & 17 & -22 \end{pmatrix}\begin{pmatrix} 24 \\ -20 \\ -6 \end{pmatrix} = \frac{1}{40}\begin{pmatrix} 24+380-204 \\ 312+140-12 \\ -72-340+132 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \\ -7 \end{pmatrix}$$

(c) Since $x + y + z = 9$ is also a linear equation in the same variables, we add this equation to the linear system. It will simplify the computations if we put this new equation first in the system. We use Gauss elimination to solve the new linear system:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 3 & 4 & 5 & r \\ 7 & 2 & 11 & s \\ 5 & 1 & 6 & t \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & r-27 \\ 0 & -5 & 4 & s-63 \\ 0 & -4 & 1 & t-45 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & r-27 \\ 0 & 0 & 14 & s-63+5(r-27) \\ 0 & 0 & 9 & t-45+4(r-27) \end{array}\right)$$

After simplifying the expressions in the last matrix, we multiply the third row with 9 and the fourth row with 14, in order to make the last row operation easier:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & r-27 \\ 0 & 0 & 14 & 5r+s-198 \\ 0 & 0 & 9 & 4r+t-153 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & r-27 \\ 0 & 0 & 126 & 9(5r+s-198) \\ 0 & 0 & 126 & 14(4r+t-153) \end{array}\right)$$

This means that the linear system has solutions if and only if the following equation holds:

$$9(5r+s-198) = 14(4r+t-153) \quad \Leftrightarrow \quad 45r + 9s - 1782 = 56r + 14t - 2142$$

This gives $-11r + 9s - 14t = -360$, or $11r - 9s + 14t = 360$. We therefore conclude that there are solutions of $A\mathbf{x} = \mathbf{b}$ with $x + y + z = 9$ if and only if $11r - 9s + 14t = 360$.

Question 5.

Since the region $D = \{(x,y) : 0 \le x,y \le 1\}$ is a square and therefore compact, f has a maximum and minimum value on D. Candidates points are the stationary points for f that are interior points of D, and boundary points (the four sides of the square). To find stationary points of f, we solve the first order conditions

$$f'_x = \sqrt{y} \cdot \frac{1}{2\sqrt{x}} - 1 = 0, \quad f'_y = \sqrt{x} \cdot \frac{1}{2\sqrt{y}} = 0$$

There are no interior points satisfying these equations, since the second equation gives $x = 0$. We look at the boundary points of D, and find the greatest and smallest value of $f(x,y) = \sqrt{xy} - x$ on each of the four sides. When $x = 0$, we have $f(0,y) = 0$ for $0 \le y \le 1$. When $y = 0$, we have $f(x,0) = -x$, and the minimum value is $f(1,0) = -1$ and the maximum value is $f(0,0) = 0$ when $0 \le x \le 1$ since the function is decreasing. When $x = 1$, we have $f(1,y) = \sqrt{y} - 1$, and the minimum value is $f(1,0) = -1$ and the maximum value is $f(1,1) = 0$ when $0 \le y \le 1$ since the function is increasing. When $y = 1$, we have $f(x,1) = \sqrt{x} - x$ for $0 \le x \le 1$, with derivative

$$(\sqrt{x} - x)' = \frac{1}{2\sqrt{x}} - 1 = \frac{1 - 2\sqrt{x}}{2\sqrt{x}}$$

The derivative is zero when $\sqrt{x} = 1/2$, or $x = 1/4$, and we see that the function is increasing for $x \le 1/4$ and decreasing for $x \ge 1/4$. This means that $f(1/4,1) = 1/2 - 1/4 = 1/4$ is the maximum value, and $f(0,1) = f(1,1) = 0$ is the minimum value. Since we have considered all four sides, we can conclude that $f_{\max} = 1/4$ and that $f_{\min} = -1$.

Question 6.

(a) The equation $y^2 = 5x^2 - x^3$ of the curve C can be written $y^2 + x^3 - 5x^2 = 0$ in standard form $g(x,y) = a$. The slope of the tangent at a point (x,y) on the curve C is given by

$$y' = -\frac{g'_x}{g'_y} = -\frac{3x^2 - 10x}{2y}$$

and $y' = -1$ gives $3x^2 - 10x = 2y$, or $y = x(3x - 10)/2$. We substitute this expression for y into the equation for C, which gives

$$\begin{aligned} y^2 + x^3 - 5x^2 &= \frac{x^2(3x-10)^2}{4} + x^2(x-5) = \frac{x^2(3x-10)^2}{4} + \frac{x^2(x-5)\cdot 4}{4} \\ &= \frac{x^2(9x^2 - 60x + 100 + 4x - 20)}{4} = \frac{x^2(9x^2 - 56x + 80)}{4} = 0 \end{aligned}$$

This means that $x = 0$ or $9x^2 - 56x + 80$. If $x = 0$, then $y = 0$, and we obtain the point $(0,0)$. If $9x^2 - 56x + 80 = 0$, then

$$x = \frac{56 \pm \sqrt{56^2 - 4 \cdot 9 \cdot 80}}{2 \cdot 9} = \frac{56 \pm 16}{18} = \frac{28 \pm 8}{9}$$

which gives $x = 4$ or $x = 20/9$. When we substitute $x = 4$ in the equation $y = x(3x - 10)/2$, we find $y = 4$, and when we substitute $x = 20/9$, we find $y = -100/27$. Therefore, we find the points $(4,4), (20/9, -100/27)$ with $(x,y) \neq (0,0)$.

(b) We use the method of Lagrange multipliers, and let $\mathcal{L}(x,y;\lambda) = x + y - \lambda(y^2 + x^3 - 5x^2)$. The Lagrange conditions are given by

$$\mathcal{L}'_x = 1 - \lambda(3x^2 - 10x) = 0, \quad \mathcal{L}'_y = 1 - \lambda(2y) = 0, \quad y^2 + x^3 - 5x^2 = 0$$

We eliminate λ from the first two equations, and this gives

$$\lambda = \frac{1}{3x^2 - 10x} = \frac{1}{2y} \quad \Rightarrow \quad 2y = 3x^2 - 10x \quad \Rightarrow \quad y = x(3x - 10)/2$$

We see that we find the same equations as in (a), hence $(x,y) = (4,4), (20/9, -100/27), (0,0)$. We substitute these points into the second equation to find λ: The point $(x,y) = (0,0)$ gives $1 - \lambda \cdot 0 = 0$, which is a contradiction. The point $(4,4)$ gives $1 = 8\lambda$, or $\lambda = 1/8$, and the point $(20/9, -100/27)$ gives $1 = \lambda \cdot (-200/27)$, or $\lambda = -27/200$. We conclude that there are two solutions of the Lagrange conditions:

$$(x,y;\lambda) = (4,4;1/8), (20/9, -100/27; -27/200)$$

with respectively $f(4,4) = 8$ and $f(20/9, -100/27) = -40/27$.

(c) In addition to the two candidate points from (b), we can also include admissible points with degenerated constraint: We solve the equations

$$g'_x(x,y) = 3x^2 - 10x = 0, \quad g'_y(x,y) = 2y = 0, \quad y^2 + x^3 - 5x^2 = 0$$

This gives the point $(x,y) = (0,0)$ with $f(0,0) = 0$. The best candidate for maximum in the Lagrange problem is therefore $f(4,4) = 8$. Notice that the level curve for $f(x,y) = x + y$

through this point is $x + y = 8$, and this is the tangent of C at the point (4,4): This is a consequence of theory, but we can also see this concretely since $y - 4 = -1(x - 4)$ gives $x + y = 8$. Next, we find the intersections of this tangent with the curve C: We substitute $y = 8 - x$ in the equation of C, and find $(8 - x)^2 + x^3 - 5x^2 = 0$, or $x^3 - 4x^2 - 16x + 64 = 0$. We know that $x = 4$ is a solution, since this is an intersection point. Polynomial division gives

$$x^3 - 4x^2 - 16x + 64 = (x - 4)(x^2 - 16) = 0 \quad \Rightarrow \quad (x - 4)(x - 4)(x + 4) = 0$$

Hence $x = -4$ gives another intersection point, with $y = 8 - (-4) = 12$, and therefore $f(-4,12) = 8 = f(4,4)$. Since C and $x + y = 8$ do not meet at a tangent in the point $(-4,12)$, there are points on the curve C close to $(-4,12)$ with $f(x,y) > 8$. It follows that the Lagrange problem has no maximum.

Question 7.

We write $x = \sqrt[3]{7 + \sqrt{50}} + \sqrt[3]{7 - \sqrt{50}} = u + v$ with $u = \sqrt[3]{7 + \sqrt{50}}$ and $v = \sqrt[3]{7 - \sqrt{50}}$. To find a polynomial $p(x)$ with x as a root, we compute

$$x^3 = (u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 = u^3 + v^3 + 3uv(u + v)$$

We have that $u^3 + v^3 = (7 + \sqrt{50}) + (7 - \sqrt{50}) = 14$, and since $(7 + \sqrt{50})(7 - \sqrt{50}) = 49 - 50 = -1$, we also have that

$$3uv = 3\,\sqrt[3]{7 + \sqrt{50}}\,\sqrt[3]{7 - \sqrt{50}} = 3\,\sqrt[3]{-1} = 3(-1) = -3$$

Hence x must be a root of the equation $x^3 = 14 - 3x$, which we can write as $x^3 + 3x - 14 = 0$. We conclude that we may choose $p(x) = x^3 + 3x - 14$. Since $p(x) = x^3 + 3x - 14$ has derivative $p'(x) = 3x^2 + 3 > 0$, it follows that p is a strictly increasing function and has exactly one real root. On the other hand, we have that $p(2) = 2^3 + 3 \cdot 2 - 14 = 8 + 6 - 14 = 0$. This means that $x = 2$ is the unique root of p, and

$$x = \sqrt[3]{7 + \sqrt{50}} + \sqrt[3]{7 - \sqrt{50}} = 2$$