Solutions: EBA29104 12/2021

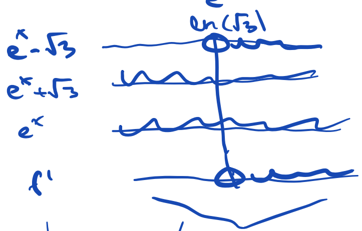
1. a) $f'(x) = e^{x} - 3e^{-x}$

$f''(x) = e^{x} + 3e^{-x} > 0$ for all $x \Rightarrow f$ <u>convex</u>

b) $f'(x) = \underline{e^{x} - 3e^{-x}} = e^{x} - \frac{3}{e^{x}} = \frac{e^{2x}-3}{e^{x}} = 0 \Rightarrow e^{2x} = 3 \Rightarrow e^{x} = \sqrt{3}$

Stat. pts: $\underline{x = \ln(\sqrt{3})}$

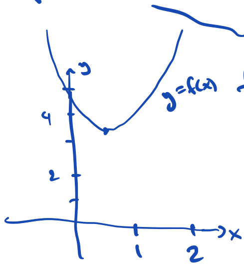
$f'(x) = \frac{(e^{x}-\sqrt{3})(e^{x}+\sqrt{3})}{e^{x}}$



$f_{min} = f(\ln(\sqrt{3})) = \sqrt{3} + \frac{3}{\sqrt{3}} = \underline{\underline{2\sqrt{3}}}$

no global max (since f is convex)

c)



$f_{min} = 2\sqrt{3} \simeq 3.4$

at $x = \ln(\sqrt{3}) \simeq 0.55$

f convex $\quad f \to \infty$ when $x \to \pm\infty$

d) $f(x) = e^{x} + ae^{-x}$

$f'(x) = e^{x} - ae^{-x}$

$f''(x) = e^{x} + ae^{x}$

$\underline{f'(x) = 0}$: $e^{x} - ae^{-x} = e^{x} - \frac{a}{e^{x}} = \frac{e^{2x}-a}{e^{x}} = 0$

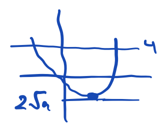
$a > 0$: $e^{x} = \sqrt{a} \Rightarrow x = \ln(\sqrt{a})$ stat. pt

$a \le 0$: no stat. pts

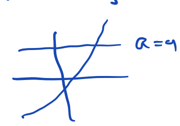
if $a > 0$, then f is convex

$\Rightarrow f_{min} = \sqrt{a} + \frac{a}{\sqrt{a}} = 2\sqrt{a}$

$\underline{a > 0}$: Note $2\sqrt{a} = f_{min} < 4 \Rightarrow f(x) = 4$ has two sol's



$\underline{a \le 0}$: $f'(x) > 0$ for all x, f increasing $\}$ $f(x) = 4$ has one solution



<u>Concl</u>: $f(x) = 4$ has two sol's

$\Updownarrow$

$a > 0$, $2\sqrt{a} < 4$

$\sqrt{a} < 2$

$\underline{\underline{0 < a < 4}}$

2. a) $\int \frac{2}{\sqrt{x}}\,dx = 2\left(2x^{1/2}\right)+C = \underline{4\sqrt{x}+C}$

b) $\int \frac{12}{4-x^2}\,dx = \int \frac{3}{2+x} + \frac{3}{2-x}\,dx = \underline{3\ln|2+x| - 3\ln|2-x| + C}$

c) $\int 9\sqrt{x}\ln(\sqrt{x})\,dx = \int 9x^{1/2}\ln(x^{1/2})\,dx = \frac{9}{2}\int x^{1/2}\ln x\,dx$

$= \frac{9}{2}\left(\frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{1/2}dx\right) = 3x\sqrt{x}\ln x - 3\left(\frac{2}{3}x^{3/2}\right)+C$

$\boxed{\begin{matrix} u=\frac{2}{3}x^{3/2} & v=\ln x \\ u'=x^{1/2} & v'=1/x \end{matrix}}$

$= \underline{3x\sqrt{x}\ln x - 2x\sqrt{x}+C}$

3. a) $\left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 0 \\ 3 & 7 & 2 & 0 & 9 \\ 5 & 12 & 3 & -3 & 16 \end{array}\right) \begin{matrix}]-3 \\ ]-5 \end{matrix} \to \left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 0 \\ 0 & 1 & -1 & -12 & 9 \\ 0 & 2 & -2 & -23 & 16 \end{array}\right)]-2$

$\to \left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 0 \\ 0 & 1 & -1 & -12 & 9 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)$

echelon form
(z free)

$x+2y+z+4w=0 \Rightarrow x = \underline{-3z+38}$

$y-z-12w=9 \Rightarrow y = \underline{z-15}$

$w = \underline{-2}$

$\Downarrow$

$(x,y,z,w) = \underline{\underline{(-3z+38, z-15, z, -2)}}$

with z free

b) $\left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 0 \\ 3 & 7 & 2 & a & 9 \\ 5 & 12 & 3 & -3 & 16 \end{array}\right) \begin{matrix}]-3 \\ ]-5 \end{matrix} \to \left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 0 \\ 0 & 1 & -1 & a-12 & 9 \\ 0 & 2 & -2 & -23 & 16 \end{array}\right)]-2$

$\to \left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 0 \\ 0 & 1 & -1 & a-12 & 9 \\ 0 & 0 & 0 & 1-2a & -2 \end{array}\right)$

$\underline{1-2a=0}$: $a=1/2 \to$ no solutions
$a\neq 1/2 \to$ inf. many solutions

consistent $\Leftrightarrow$ $\underline{\underline{a\neq 1/2}}$

c) $\left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 2 \\ 3 & 7 & 2 & a & 1 \\ 5 & 12 & 3 & -3 & 0 \end{array}\right) \begin{matrix}]-3 \\ \\ ]-5 \end{matrix} \to \left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 2 \\ 0 & 1 & -1 & a-12 & -5 \\ 0 & 2 & -2 & -23 & -10 \end{array}\right)]-2$

$\to \left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 2 \\ 0 & 1 & -1 & a-12 & -5 \\ 0 & 0 & 0 & 1-2a & 0 \end{array}\right)$

consistent for all values of a, with one or two free variables

$a = 1/2$: z, w free

$a \neq 1/2$: z free, $w = 0$

in all cases, we may take $z = w = 0$

$\Downarrow$

$y = -5$

$x + 2y = 2 \Rightarrow x = 2 + 10 = 12$

$\underline{w} = 12\underline{v_1} - 5\underline{v_2}$

4.

a) $f'_x = 2x - 2xy^2 = 2x(1-y^2) = 0$ (1) $x = 0$ or $y = \pm 1$, and

$f'_y = 2y - x^2 \cdot 2y = 2y(1-x^2) = 0$ (2) $y = 0$ or $x = \pm 1$

$\Downarrow$

Stat. pts: $(0,0)$, $(\pm 1, \pm 1)$

$H(f) = \begin{pmatrix} 2-2y^2 & -4xy \\ -4xy & 2-2x^2 \end{pmatrix}$

$H(f)(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ det $= 4$, tr $= 4$ $\Rightarrow$ $(0,0)$ local min for f

$H(f)(\pm 1, \pm 1) = \begin{pmatrix} 0 & \pm 4 \\ \pm 4 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix}$ if $x = y$ or $= \begin{pmatrix} 0 & +4 \\ +4 & 0 \end{pmatrix}$ if $x = -y$

In all cases: det $= 0 - 16 = -16$

$(\pm 1, \pm 1)$ saddle pt

b) no global max since no local max

global min: $f(0,0) = 0$ best candidate

but $f(3,3) = 9 + 9 - 81 = -63 < 0$

$\Downarrow$

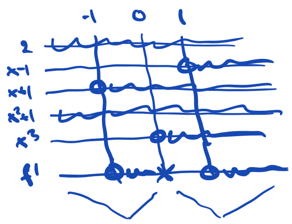
no global min

c) min $f(x,y) = x^2+y^2 - x^2y^2$ when $xy=1$

$\underline{y=1/x}$: $f = x^2 + 1/x^2 - x^2 \cdot 1/x^2 = x^2 + 1/x^2 - 1$

$f' = 2x - 2x^{-3} = 2x - \frac{2}{x^3} = \frac{2x^4-2}{x^3} = \frac{2(x^4-1)}{x^3}$

$= \frac{2(x-1)(x+1)(x^2+1)}{x^3}$



$\underline{\text{cand. for min:}}$

$x=-1$, $y=1/x=-1$

$\Rightarrow f(-1,-1) = \underline{1}$

$x=1$, $y=1/x=1$

$\Rightarrow f(1,1) = \underline{1}$

$\Downarrow$

$f_{min} = \underline{\underline{1}}$ at $(1,1), (-1,-1)$

$\underline{\text{Alt:}}$ Lagrange $\quad L = x^2+y^2-x^2y^2 - \lambda(xy-1)$

$L'_x = 2x(1-y^2) - \lambda y = 0 \qquad \lambda y = 2x(1-y^2) \quad | \cdot x \qquad \lambda xy = \lambda = 2x^2(1-y^2)$

$L'_y = 2y(1-x^2) - \lambda x = 0 \qquad \lambda x = 2y(1-x^2) \quad | \cdot y \qquad \lambda xy = \lambda = 2y^2(1-x^2)$

$xy = 1$

$\Rightarrow 2x^2(1-y^2) = 2y^2(1-x^2)$

$2x^2 - \cancel{2x^2y^2} = 2y^2 - \cancel{2x^2y^2}$

$x^2 = y^2$

$x = y$ or $x = -y$

$\underline{\text{cand. pts:}}$

$(x,y;\lambda) = (1,1;0)$, $\underline{(-1,-1;0)}$ $\Leftarrow$ $\underline{xy=1}$: $x^2=1$, $x = \pm 1$ $\qquad -x^2 = 1$ impossible

(as above) $\qquad$ use sign diagram above to see that these are $\underline{\text{min. pts}}$

d) $f^*(a) = \min\left\{x^2+y^2-x^2y^2 \text{ when } xy=a\right\}$

$\underline{\text{From theory}}$: $\frac{df^*(a)}{da} = \lambda = 0$ at $\underline{a=1}$

$\Rightarrow f^*(a) \simeq f^*(1) + 0 \cdot \Delta a = \underline{\underline{1}}$

for a close to $a=1$.

$\underline{5.}$ a) max $f(x,y) = x+y$ when $\underline{g(x,y) \le a}$

marked area

since $-2 \le x, y \le 2$ for all pts in $D = \{(x,y) : g(x,y) \le a\}$

(from figure)

D is $\underline{\text{compact}}$ $\Rightarrow$ $\underline{\text{there is a}}$ $\underline{\text{max}}$ $\Rightarrow$ either $\underline{\text{boundary pt}}$ (blue curve)

or $\underline{\text{interior stat. pt}}$

$\left.\begin{array}{l} f'_x = 1 = 0 \\ f'_y = 1 = 0 \end{array}\right\}$ no sol's $\Rightarrow$ no interior stat. pts

$\Downarrow$

$\underline{\underline{\text{there is a max on the blue curve}}}$

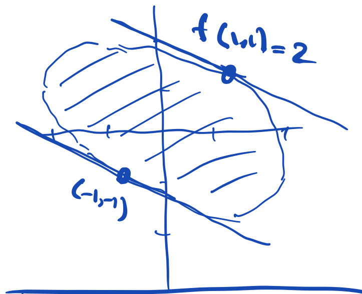
b) Level curves of f:

$f(x,y) = x+y = c$

$\Downarrow$

$y = c - x$

Lines with slope -1



two straight lines with slope -1 that meet the blue curve at a tangent $\Rightarrow$ min (lower line), and max (upper line).

Read off from the figure that the upper line meet the blue curve at approx $(1,1) \Rightarrow f_{max} = 1+1 = 2$