Solutions: EBA 29104 2022-05-23

1.

$$\left(\begin{array}{cccc|c} 2 & -6 & 4 & 6 & 8 \\ 3 & a & 7 & 2 & 7 \\ 1 & -2 & 1 & 10 & 5 \end{array}\right) \cdot(-1) \rightarrow \left(\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 3 \\ 3 & a & 7 & 2 & 7 \\ 1 & -2 & 1 & 10 & 5 \end{array}\right) \begin{array}{l} \cdot(-3) \\ \cdot(-1) \end{array}$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 3 \\ 0 & a+12 & -2 & 14 & -2 \\ 0 & 2 & -2 & 14 & 2 \end{array}\right)$$

a) $a=-12$:

$$\left(\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 3 \\ 0 & 0 & -2 & 14 & -2 \\ 0 & 2 & -2 & 14 & 2 \end{array}\right) \updownarrow \rightarrow \left(\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 3 \\ 0 & 2 & -2 & 14 & 2 \\ 0 & 0 & -2 & 14 & -2 \end{array}\right)$$ echelon form

$$\begin{aligned} x-4y+3z-4w &= 3 \\ 2y-2z+14w &= 2 \\ -2z+14w &= -2 \end{aligned}$$

w free

$x = 4\cdot 2-3(7w+1)+4w+3 \Rightarrow x = \underline{8-17w}$

$2y = 2(7w+1)-14w+2 = 4 \Rightarrow \underline{y=2}$

$-2z = -14w-2 \Rightarrow z = \underline{7w+1}$

Solution: $(x,y,z,w) = \underline{(8-17w,\ 2,\ 1+7w,\ w)}$ with w free

b) Continue Gaussian process with general a:

$$\left(\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 3 \\ 0 & a+12 & -2 & 14 & -2 \\ 0 & 2 & -2 & 14 & 2 \end{array}\right) \updownarrow \rightarrow \left(\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 3 \\ 0 & 2 & -2 & 14 & 2 \\ 0 & a+12 & -2 & 14 & -2 \end{array}\right) \cdot \tfrac{1}{2}$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 3 \\ 0 & 1 & -1 & 7 & 1 \\ 0 & a+12 & -2 & 14 & -2 \end{array}\right) \cdot(-(a+12)) \rightarrow \left(\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 3 \\ 0 & 1 & -1 & 7 & 1 \\ 0 & 0 & a+10 & * & ** \end{array}\right)$$

where $* = 14-7(a+12) = -70-7a$
$= -7(a+10)$
$** = -2-(a+12) = -14-a$

For the system to have no solutions, we need $a+10,\ * = 0$ but $** \neq 0$.

Hence: $$\left.\begin{aligned} a+10 &= 0 \\ -7(a+10) &= 0 \\ -14-a &\neq 0 \end{aligned}\right\}$$ $a=-10$ means $a+10,\ *=0$ but then $-14-a = -4 \neq 0$

$a=-10$:

$$\left(\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 3 \\ 0 & 1 & -1 & 7 & 1 \\ 0 & 0 & 0 & 0 & -4 \end{array}\right)$$

Conclusion: no solutions $\Leftrightarrow$ $\underline{\underline{a=-10}}$

2.

a) $$\int_0^1 6\sqrt{x} - 11x\sqrt[5]{x}\,dx = \int_0^1 6x^{1/2} - 11x^{6/5}\,dx$$
$$= \left[6\cdot\left(\frac{2}{3}x^{3/2}\right) - 11\cdot\left(\frac{5}{11}x^{11/5}\right) + C\right]_0^1 = \left[4x^{3/2} - 5x^{11/5} + C\right]_0^1$$
$$= \left[4x\sqrt{x} - 5x^2\sqrt[5]{x}\right]_0^1 = (4-5) - 0 = \underline{\underline{-1}}$$

b) $$\int \frac{21-x}{9-x^2}\,dx = \int \frac{3}{3-x} + \frac{4}{3+x}\,dx = 3\ln|3-x|\cdot\frac{1}{(-1)} + 4\ln|3+x|\cdot\frac{1}{1} + C$$
$$= \underline{4\ln|3+x| - 3\ln|3-x| + C}$$

$$\frac{21-x}{9-x^2} = \frac{A}{3-x} + \frac{B}{3+x} \quad \Big|\cdot(3-x)(3+x)$$
$$21-x = A(3+x) + B(3-x)$$
$$21-x = (3A+3B) + x(A-B)$$
$$\Downarrow$$
$3A+3B = 21$
$A-B = -1$

$A+B = 7$
$A-B = -1$
$2A = 6$
$A = \underline{3}$ $\quad B = 7 - A = \underline{4}$

c) $$\int \frac{1}{1-\sqrt{x}}\,dx = \int \frac{1}{u}\cdot(-2\sqrt{x})\,du = \int \frac{-2(1-u)}{u}\,du$$

$$\boxed{\begin{array}{l} u = 1-\sqrt{x} \\ du = -\frac{1}{2\sqrt{x}}\,dx \end{array}}$$
$\sqrt{x} = 1-u$
$dx = -2\sqrt{x}\,du$

$$= \int \frac{-2+2u}{u}\,du = \int 2 - \frac{2}{u}\,du = 2u - 2\ln|u| + C$$
$$= \underline{2(1-\sqrt{x}) - 2\ln|1-\sqrt{x}| + C}$$

d) $$\int_0^1 f'(x)\,dx = f(1) - f(0) \Rightarrow f(1) - f(0) = -A,$$ where

A = area between graph of $f'(x)$ and x-axis on interval $[0,1]$

$\approx$ ca 4 sq. $= 4\cdot\frac{1}{16} = \frac{1}{4} = 0.25$

Since $f(x)$ is an antiderivative of $f'(x)$ by defn.

__Conclusion:__ $f(1) - f(0) = -A$

$\approx \underline{\underline{-0.25}}$

e) max/min $f(x)$:

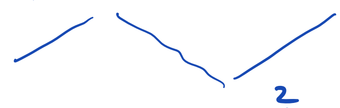
<u>Candidate pts:</u>

i) Boundary pts. $\underline{x=-1}$, $\underline{x=2}$

ii) Stationary pts. $\underline{x=0}$, $\underline{x=1}$ $(f'(x)=0)$

<u>Sign diagram of $f'(x)$:</u>

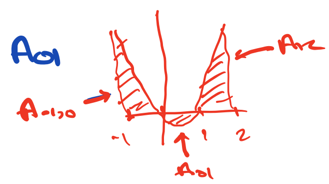
$$\begin{array}{c|ccccccc} & -1 & & 0 & & 1 & & 2 \\ f'(x) & & + & 0 & - & 0 & + & \end{array}$$



<u>Possible max:</u> $x=0$, $x=2$

$$f(2)-f(0)=\int_0^2 f'(x)\,dx = -A_{01}+A_{12}>0$$

since the area $A_{12}>A_{01}$ in the figure



$\Downarrow$

$\boxed{f(2)>f(0)}$

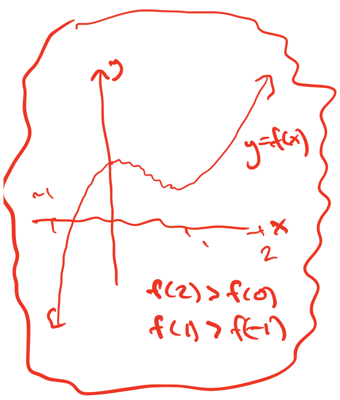
<u>Possible min:</u> $x=-1$, $x=1$

$$f(1)-f(-1)=\int_{-1}^{1} f'(x)\,dx = A_{-1,0}-A_{01}>0$$

since the area $A_{-1,0}>A_{01}$

$\Downarrow$

$\boxed{f(1)>f(-1)}$



<u>Conclusion:</u> $x=2$ is the <u>max.</u> pt. for f

$x=-1$ is the <u>min</u> pt. for f

3. $A=\begin{pmatrix} a & 1 & 2 \\ 1 & a & 1 \\ 2 & 1 & a \end{pmatrix}$

$$|A|=\begin{vmatrix} a & 1 & 2 \\ 1 & a & 1 \\ 2 & 1 & a \end{vmatrix} = a(a^2-1)-1\cdot(a-2)+2(1-2a)$$
$$= a^3-a-a+2+2-4a=\underline{a^3-6a+4}$$

(computing $|A|$ for any a)

a) $|A|=4\neq 0$ when $\underline{a=0}$

$\Downarrow$

$$A^{-1}=\frac{1}{|A|}\cdot\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{4}\begin{pmatrix} -1 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 2 & -1 \end{pmatrix}^T = \underline{\underline{\frac{1}{4}\begin{pmatrix} -1 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 2 & -1 \end{pmatrix}}}$$

$A=\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ $C_{11}=-1$ $C_{12}=2$ $C_{13}=1$

$C_{22}=-4$ $C_{23}=2$

$C_{33}=-1$

A symmetric $\Rightarrow$ adj(A) symm.

b) $|A| = a^3 - 6a + 4$: See that $|A| = a^3 - 6a + 4 = 2^3 - 6\cdot 2 + 4 = 8 - 12 + 4 = 0$ when $\underline{a = 2}$, i.e. $a - 2$ is a factor in $|A|$

$$a^3 - 6a + 4 : a - 2 = a^2 + 2a - 2$$
$$\begin{array}{l} a^3 - 2a^2 \\ \hline 2a^2 - 6a + 4 \\ 2a^2 - 4a \\ \hline -2a + 4 \\ -2a + 4 \\ \hline 0 \end{array}$$

$\Downarrow$

$|A| = \underline{a^3 - 6a + 4} = (a-2)(a^2 + 2a - 2)$

$= \underline{(a-2)(a - (-1+\sqrt{3}))(a - (-1-\sqrt{3}))}$

$$a^2 + 2a - 2 = 0$$
$$a = \frac{-2 \pm \sqrt{4 - 4(-2)}}{2} = \frac{-2 \pm \sqrt{12}}{2} = \frac{-2 \pm \sqrt{3}\cdot\sqrt{4}}{2} = -1 \pm \sqrt{3}$$

$|A| = 0$ for $\underline{a = 2}$, $\underline{a = -1 \pm \sqrt{3}}$

c) When $a \neq 2, -1 \pm \sqrt{3}$, we have $|A| \neq 0 \Rightarrow$ $A\underline{x} = \underline{0}$ gives $\underline{x} = A^{-1}\cdot\underline{0} = 0$

($\underline{x} = \underline{0}$ is the only solution)

We are left with the cases $\underline{a = 2}$, $\underline{a = -1 \pm \sqrt{3}}$, and $a = 2$ gives the simplest calculation:

$\underline{a = 2}$: $\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \cdot \underline{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Know that $|A| = 0$, so there are inf. many sol's since $\underline{b} = \underline{0}$

Solve using Gauss:

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right) \begin{array}{l} \\ \cdot(-2) \\ \cdot(-2) \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{array}\right) \cdot(-1) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\underline{x} + 2y + z = 0$
$-3y = 0$
(z free)

$\Downarrow$

$\underline{y = 0}$
$x = -2\cdot 0 - z = \underline{-z}$

Result:

$(x, y, z) = \underline{(-z, 0, z)}$ with z free

For example, when $\underline{z = 1}$, we get

$(x, y, z) = \underline{(-1, 0, 1)}$: $\quad A\cdot\underline{x} = \underline{0}$ when $\underline{(x, y, z) = (-1, 0, 1)}$ og $\underline{a = 2}$

There are many other solutions.

4. $f(x,y) = x^2y - 5xy^2 + xy^3$

a) $f'_x = 2xy - 5y^2 + y^3 = y(2x - 5y + y^2) = 0$

$f'_y = x^2 - 10xy + 3xy^2 = x(x - 10y + 3y^2) = 0$

$$\boxed{\begin{array}{lll} y=0 & \text{or} & 2x-5y+y^2=0 \\ x=0 & \text{or} & x-10y+3y^2=0 \end{array}}$$

i) $x=0, y=0$: $(x,y) = (0,0)$

ii) $y=0$, $x - 10y + 3y^2 = 0$: $x=0 \to (x,y) = (0,0)$

iii) $x=0$, $2x - 5y + y^2 = 0$: $\left.\begin{array}{l} y^2 - 5y = y(y-5) = 0 \\ y=0 \text{ or } y=5 \end{array}\right\}$ $(x,y) = (0,0), (0,5)$

iv) $\left.\begin{array}{l} 2x - 5y + y^2 = 0 \\ x - 10y + 3y^2 = 0 \end{array}\right\}$

$x = 10y - 3y^2$

$2(10y - 3y^2) - 5y + y^2 = 0$

$-5y^2 + 15y = 0$

$-5y(y-3) = 0$

$y=0$ or $y=3$

$x=0$ $\quad y = 30 - 27 = 3$

$(x,y) = (0,0), (3,3)$

Conclusion: Stat. pts. for f are: $(0,0)$, $(0,5)$, $(3,3)$

b) $H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} = \begin{pmatrix} 2y & 2x - 10y + 3y^2 \\ 2x - 10y + 3y^2 & -10x + 6xy \end{pmatrix}$

$(0,5)$: $H(f)(0,5) = \begin{pmatrix} 10 & 25 \\ 25 & 0 \end{pmatrix}$ $\quad \det = 10 \cdot 0 - 25^2 = -625 < 0$

$(0,5)$ is a saddle pt

$(3,3)$: $H(f)(3,3) = \begin{pmatrix} 6 & 3 \\ 3 & 24 \end{pmatrix}$ $\quad \det = 6 \cdot 24 - 3 \cdot 3 = 144 - 9 = 135 > 0$

$\mathrm{tr} = 6 + 24 = 30 > 0$

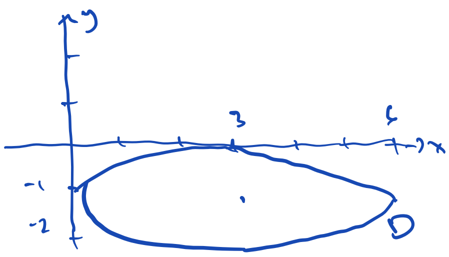
$(3,3)$ is a local min.

5. max $f(x,y) = x+3y$ when $x^2-6x+9y^2+18y+9=0$

a) $D:\ x^2-6x+9y^2+18y+9=0$

$x^2-6x+9+9(y^2+2y+1)+\cancel{9} = \cancel{9}+9 \quad |:9$

$\frac{(x-3)^2}{9}+\frac{(y+1)^2}{1}=1$ ellipse, center $(3,-1)$, half-axes $a=\sqrt{9}=3$, $b=\sqrt{1}=1$



D is bounded since

$0 \le x \le 6$

$-2 \le y \le 0$

b) $L = x+3y-\lambda(x^2-6x+9y^2+18y+9)$

$$\boxed{\begin{aligned} L'_x &= 1-\lambda(2x-6)=0 \\ L'_y &= 3-\lambda\cdot(18y+18)=0 \\ & x^2-6x+9y^2+18y+9=0 \end{aligned}}$$

Lagrange-conditions

There is a max by the extreme value thm. since D is bounded.
Since D is an ellipse, there is a unique tangent at every pt, so no adm. pts with degenerate constr.

$\Downarrow$

max = ordinary candidate pt with greatest value

$\lambda = \frac{1}{2x-6} = \frac{3}{18y+18}$

$18y+18 = 3(2x-6)$

$18(y+1) = 6(x-3)$

$\Rightarrow x-3 = 3\cdot(y+1)$

Substitute in the constraint: $\frac{(x-3)^2}{9}+\frac{(y+1)^2}{1}=1$

$\frac{3^2(y+1)^2}{9}+(y+1)^2=1$

$2(y+1)^2=1 \Rightarrow (y+1)^2=1/2$

$\lambda = \frac{1}{2(x-3)}$

$= \frac{1}{2} \cdot \frac{1}{3(\pm\sqrt{1/2})}$

$= \pm \frac{1}{6\sqrt{1/2}}$

$x - 3 = 3 \cdot (\pm\sqrt{1/2})$

$x = 3 \pm 3\sqrt{1/2}$

$\Leftarrow \begin{cases} y+1 = \pm\sqrt{1/2} \\ y = -1 \pm \sqrt{1/2} \end{cases}$

Candidate pts: $(x, y; \lambda) = \left(3 + 3\sqrt{1/2}, -1 + \sqrt{1/2}, \frac{1}{6\sqrt{1/2}}\right)$, $f = 6\sqrt{1/2}$

$\left(3 - 3\sqrt{1/2}, -1 - \sqrt{1/2}; \frac{-1}{6\sqrt{1/2}}\right)$ $f = -6\sqrt{1/2}$

$f_{max} = 6 \cdot \sqrt{1/2} = \frac{6 \cdot \sqrt{1} \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$

at $\left(3 + 3\sqrt{1/2}, -1 + \sqrt{1/2}; \frac{1}{6\sqrt{1/2}}\right)$

6. a) Write constraint as $x(x^2+y^2) - (x^2-y^2) = 0$

$g(x,y) = x^3 + xy^2 - x^2 + y^2 = 0$

Degenerated constraint:

(1) $g'_x = 3x^2 + y^2 - 2x = 0$

(2) $g'_y = 2xy + 2y = 0 \Rightarrow 2y(x+1) = 0 \Rightarrow y = 0$ or $x = -1$

$y = 0$: (1) gives $3x^2 - 2x = 0$
$x(3x-2) = 0$
$x = 0, x = 2/3$ $\Bigg\} \Rightarrow (0,0), (2/3, 0)$

$x = -1$: (1) gives $3 + y^2 + 2 = 0$
$5 + y^2 = 0$
impossible $\Bigg\}$ no pts

Among the pts $(0,0)$, $(2/3, 0)$, only $(0,0)$ is adm: $g(0,0) = 0$ but $g(2/3, 0) \neq 0$

Concl: One adm pt $(0,0)$ with degenerated constraints

b) max $f(x,y) = y$ when $x(x^2+y^2) = x^2-y^2$

Level curves for f:

$f(x,y) = a$

$y = a$

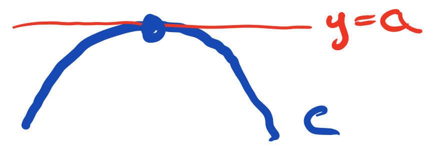
horizontal straight lines

C : $x(x^2+y^2) = x^2-y^2$

has horizontal tangent

$\Uparrow$

the level curve $y=a$ meets C at a tangent



$\Uparrow$

the pts (x,y) on C (adm. pts) that satisfy FOC $\nabla f = \lambda \cdot \nabla g$ for some λ

$\Uparrow$

ordinary candidate pts, which satisfy the Lagrange-conditions FOC + C.