Exam	EBA 2910 Mathematics for Business Analytics
Date	May 23rd 2022 at 0900 - 1400

The exam paper consists of 16 problems with equal weight. You must justify your answers.

Question 1.
We consider a linear system with parameter a, given in matrix form as

$$\begin{pmatrix} 2 & -6 & 4 & 6 \\ 3 & a & 7 & 2 \\ 1 & -2 & 1 & 10 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \\ 5 \end{pmatrix}$$

(a) **(6p)** Solve the linear system when $a = -12$.
(b) **(6p)** Determine the values of a (if any) such that the linear system has no solutions.

Question 2.
Compute these integrals:

a) **(6p)** $\displaystyle\int_0^1 6\sqrt{x} - 11\,x\sqrt[5]{x}\;\mathrm{d}x$ b) **(6p)** $\displaystyle\int \frac{21-x}{9-x^2}\;\mathrm{d}x$ c) **(6p)** $\displaystyle\int \frac{1}{1-\sqrt{x}}\;\mathrm{d}x$

The figure below shows the graph of $f'(x)$ for a function f with domain of definition $D_f = [-1,2]$.

d) **(6p)** Estimate the value of $f(1) - f(0)$. State the result you use.
e) **(6p)** Find the x-coordinates of the maximum and minimum points of f (if they exist).

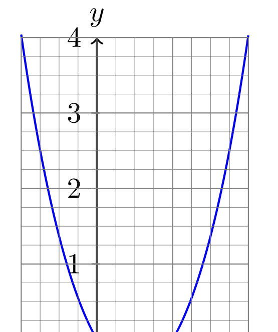

FIGURE 1. The graph of $f'(x)$

Question 3.
Let the matrix A be given by

$$A = \begin{pmatrix} a & 1 & 2 \\ 1 & a & 1 \\ 2 & 1 & a \end{pmatrix}$$

(a) **(6p)** Find A^{-1} when $a = 0$.
(b) **(6p)** Compute the determinant $|A|$ for an arbitrary value of a, and determine when $|A| = 0$.
(c) **(6p)** Find a vector $\mathbf{x} = (x,y,z) \neq (0,0,0) = \mathbf{0}$ and a value of a such that $A \cdot \mathbf{x} = \mathbf{0}$.

Question 4.
We consider the function f given by $f(x,y) = x^2y - 5xy^2 + xy^3$.

(a) **(6p)** Find the three stationary points of f.
(b) **(6p)** Compute the Hessian of f, and classify the stationary points $(x,y) \neq (0,0)$.

Question 5.
We consider the Lagrange problem $\max f(x,y) = x + 3y$ when $x^2 - 6x + 9y^2 + 18y + 9 = 0$.

(a) **(6p)** Make a sketch of $D = \{(x,y) : x^2 - 6x + 9y^2 + 18y + 9 = 0\}$. Is it a bounded set?
(b) **(6p)** Solve the Lagrange problem, and find its maximum value.

Question 6.
We consider the Lagrange problem $\max f(x,y) = y$ when $x(x^2 + y^2) = x^2 - y^2$.

(a) **(6p)** Determine whether there are admissible points with degenerated constraint.
(b) **(6p)** Explain why the points on the curve $C = \{(x,y) : x(x^2 + y^2) = x^2 - y^2\}$ with horizontal tangent are exactly the points that satisfy the Lagrange conditions.

Formula Sheet

Financial mathematics

Geometric series.
A finite geometric series has sum

$$S_n = a_1 \cdot \frac{1-k^n}{1-k} = a_1 \cdot \frac{k^n - 1}{k-1}$$

and an infinite geometric series has sum

$$S = a_1 \cdot \frac{1}{1-k} \quad \text{when } |k| < 1$$

Present values.
The present value K_0 of a payment K is given by

$$K_0 = \frac{K_n}{(1+r)^n} \quad \text{and} \quad K_0 = \frac{K_n}{e^{rn}}$$

using discrete and continuous compounding.

Integration

Integration techniques.

a) Integration by parts:

$$\int u'v \, \mathrm{d}x = uv - \int uv' \, \mathrm{d}x$$

b) Substitution:

$$\int f(u)u' \, \mathrm{d}x = \int f(u) \, \mathrm{d}u$$

c) Partial fractions:

$$\int \frac{px+q}{(x-a)(x-b)} \, \mathrm{d}x = \int \left(\frac{A}{x-a} + \frac{B}{x-b} \right) \mathrm{d}x$$

Area.
The area of the region bounded by $a \leq x \leq b$ and $f(x) \leq y \leq g(x)$ is given by

$$A = \int_a^b (g(x) - f(x)) \, \mathrm{d}x$$

Linear algebra

Cramer's rule.
A linear system $A\mathbf{x} = \mathbf{b}$ where $|A| \neq 0$ has a unique solution given by

$$x_1 = \frac{|A_1(\mathbf{b})|}{|A|} \quad x_2 = \frac{|A_2(\mathbf{b})|}{|A|} \quad \ldots \quad x_n = \frac{|A_n(\mathbf{b})|}{|A|}$$

where $A_i(\mathbf{b})$ is the matrix obtained by replacing column i of A by $\mathbf{b}$.

Functions of two variables

Second derivative test.
A stationary point (x^*, y^*) of the function $f(x,y)$ is a

a) local minimum if $A > 0$ and $AC - B^2 > 0$
b) local maximum if $A < 0$ and $AC - B^2 > 0$
c) saddle point if $AC - B^2 < 0$

when $H(f)(x^*, y^*) = \left(\begin{smallmatrix} A & B \\ B & C \end{smallmatrix} \right)$.

Level curves.
The slope $y' = \mathrm{d}y/\mathrm{d}x$ of the tangent line to the level curve $f(x,y) = c$ is given by

$$y' = -\frac{f'_x}{f'_y}$$

Method of Lagrange multipliers.
The Lagrange conditions for the problem

$$\max / \min f(x,y) \text{ when } g(x,y) = a$$

is given by

$$\mathcal{L}'_x = 0, \ \mathcal{L}'_y = 0, \ g(x,y) = a$$

An admissible point has degenerated constraint if

$$g'_x = 0, \ g'_y = 0$$