

The exam paper consists of 16 problems with equal weight. You must justify your answers.

Question 1.

We consider a linear system $A\mathbf{x} = \mathbf{b}$ with parameter a , where

$$A = \begin{pmatrix} 2 & 9 & 5 & -9 \\ 4 & a & 10 & -18 \\ 1 & 5 & 2 & -2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 7 \\ 22 \\ 3 \end{pmatrix}$$

- (a) **(6p)** Solve the linear system when $a = 22$.
(b) **(6p)** Determine the values of a such that the linear system is consistent.

Question 2.

Compute these integrals:

a) **(6p)** $\int_0^1 7x\sqrt[3]{x} \, dx$ b) **(6p)** $\int_0^1 \frac{9}{8+7x-x^2} \, dx$ c) **(6p)** $\int \frac{1}{x-\sqrt{x}} \, dx$

The figure below shows the graph of $f'(x)$ for a function f with domain of definition $D_f = [-1, 2]$.

- d) **(6p)** Estimate the value of $f(1) - f(0)$. Justify your answer.
e) **(6p)** Find the x -coordinates of the maximum and minimum points of f , if they exist.

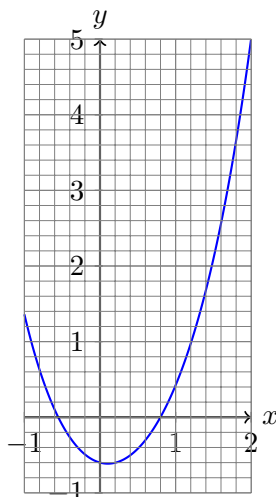


FIGURE 1. The graph of $f'(x)$

Let $I(x) = 100 \cdot 1.06^x$ be the net cash flow after x years (in million NOK per year) obtained by renting out a property. We think of this as a continuous cash flow, and use continuous discounting with discount rate $r = 10\%$ to compute net present values.

- f) **(6p)** Find the total net present value of this cash flow in the first 10 years.

Question 3.

Let the matrix A be given by

$$A = \begin{pmatrix} a & 2 & 3 \\ 2 & a & 3 \\ 2 & 3 & a \end{pmatrix}$$

- (a) **(6p)** Find A^{-1} when $a = 1$.
- (b) **(6p)** Compute the determinant $|A|$ for an arbitrary value of a , and determine when $|A| = 0$.

Question 4.

We consider the function f given by $f(x,y) = (4 - x^2)(9 - y^2)$.

- (a) **(6p)** Find the stationary points of f .
- (b) **(6p)** Compute the Hessian of f , and classify the stationary points.
- (c) **(6p)** Find the maximum and minimum values of f , if they exist.

Question 5.

We consider the Lagrange problem $\max f(x,y) = 2x + 4y$ when $2x^2 + 12x + 3y^2 - 24y = 30$.

- (a) **(6p)** Make a sketch of $D = \{(x,y) : 2x^2 + 12x + 3y^2 - 24y = 30\}$. Is it a bounded set?
- (b) **(6p)** Determine the points in D (if any) where the tangent has slope $y' = -1/2$.
- (c) **(6p)** Solve the Lagrange problem, and find its maximum value if it exists.

Formula Sheet

FINANCIAL MATHEMATICS

Geometric series.

A finite geometric series has sum

$$S_n = a_1 \cdot \frac{1 - k^n}{1 - k} = a_1 \cdot \frac{k^n - 1}{k - 1}$$

and an infinite geometric series has sum

$$S = a_1 \cdot \frac{1}{1 - k} \quad \text{when } |k| < 1$$

Present values.

The present value K_0 of a payment K is given by

$$K_0 = \frac{K_n}{(1 + r)^n} \quad \text{and} \quad K_0 = \frac{K_n}{e^{rn}}$$

using discrete and continuous compounding.

INTEGRATION

Integration techniques.

a) Integration by parts:

$$\int u'v \, dx = uv - \int uv' \, dx$$

b) Substitution:

$$\int f(u)u' \, dx = \int f(u) \, du$$

c) Partial fractions:

$$\begin{aligned} \int \frac{px + q}{(x - a)(x - b)} \, dx \\ = \int \left(\frac{A}{x - a} + \frac{B}{x - b} \right) \, dx \end{aligned}$$

Area.

The area of the region bounded by $a \leq x \leq b$ and $f(x) \leq y \leq g(x)$ is given by

$$A = \int_a^b (g(x) - f(x)) \, dx$$

LINEAR ALGEBRA

Cramer's rule.

A linear system $A\mathbf{x} = \mathbf{b}$ where $|A| \neq 0$ has a unique solution given by

$$x_1 = \frac{|A_1(\mathbf{b})|}{|A|} \quad x_2 = \frac{|A_2(\mathbf{b})|}{|A|} \quad \dots \quad x_n = \frac{|A_n(\mathbf{b})|}{|A|}$$

where $A_i(\mathbf{b})$ is the matrix obtained by replacing column i of A by \mathbf{b} .

FUNCTIONS OF TWO VARIABLES

Second derivative test.

A stationary point (x^*, y^*) of the function $f(x, y)$ is a

- a) local minimum if $A > 0$ and $AC - B^2 > 0$
- b) local maximum if $A < 0$ and $AC - B^2 > 0$
- c) saddle point if $AC - B^2 < 0$

when $H(f)(x^*, y^*) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$.

Level curves.

The slope $y' = dy/dx$ of the tangent line to the level curve $f(x, y) = c$ is given by

$$y' = -\frac{f'_x}{f'_y}$$

Method of Lagrange multipliers.

The Lagrange conditions for the problem

$$\max / \min f(x, y) \quad \text{when } g(x, y) = a$$

is given by

$$\mathcal{L}'_x = 0, \quad \mathcal{L}'_y = 0, \quad g(x, y) = a$$

An admissible point has degenerated constraint if

$$g'_x = 0, \quad g'_y = 0$$