Key Problems

Problem 1.
Compute the definite integrals, and show each of them as an area in a figure:

a) $\int_0^4 3 \, \mathrm{d}x$ b) $\int_0^8 (10 + 3x) \, \mathrm{d}x$

Problem 2.
Compute the indefinite integrals:

a) $\int x^2 \, \mathrm{d}x$ b) $\int (8x^3 - 12x^2) \, \mathrm{d}x$ c) $\int (e^x - 6x) \, \mathrm{d}x$ d) $\int (x^2/3 - x^3/2) \, \mathrm{d}x$

Problem 3.
Find a function $f(x)$ with the given derivative and domain of definition:

a) $f'(x) = 2$, $D_f = (-\infty, \infty)$ b) $f'(x) = 2x$, $D_f = (-\infty, \infty)$ c) $f'(x) = 6x^2$, $D_f = (-\infty, \infty)$
d) $f'(x) = 1/x$, $D_f = (0, \infty)$ e) $f'(x) = 1/x$, $D_f = (-\infty, 0)$ f) $f'(x) = 1/x$, $D_f = \{x : x \neq 0\}$

Problem 4.
Find a function $f(x)$ with the given properties:

a) $\int f(x) \, \mathrm{d}x = 2 + \mathcal{C}$ b) $\int f(x) \, \mathrm{d}x = 2x + \mathcal{C}$ c) $\int f(x) \, \mathrm{d}x = 6x^2 + \mathcal{C}$ d) $\int f(x) \, \mathrm{d}x = xe^{2x} + \mathcal{C}$
e) $\int 2 \, \mathrm{d}x = f(x) + \mathcal{C}$ f) $\int 2x \, \mathrm{d}x = f(x) + \mathcal{C}$ g) $\int 6x^2 \, \mathrm{d}x = f(x) + \mathcal{C}$ h) $\int xe^{2x} \, \mathrm{d}x = f(x) + \mathcal{C}$

Problem 5.
Determine constants A and B such that

$$\int \frac{(A + Bx) \cdot e^{2x}}{2\sqrt{x}} \, \mathrm{d}x = \sqrt{x} \cdot e^{2x} + \mathcal{C}$$

Problem 6.
Compute the indefinite integrals:

a) $\int x^{-3} \, \mathrm{d}x$ b) $\int \sqrt{x} \, \mathrm{d}x$ c) $\int x\sqrt{x} \, \mathrm{d}x$ d) $\int 1/x \, \mathrm{d}x$ e) $\int 1/x^2 \, \mathrm{d}x$
f) $\int (x - 2x^3) \, \mathrm{d}x$ g) $\int x(1 - 2x) \, \mathrm{d}x$ h) $x \int (1 - 2x) \, \mathrm{d}x$ i) $\int (x + 1)^2 \, \mathrm{d}x$ j) $\int (x + 1)^7 \, \mathrm{d}x$

Problem 7.
Compute the indefinite integrals:

a) $\displaystyle\int \frac{1-3x^2}{x^2}\,\mathrm{d}x$ b) $\displaystyle\int \frac{x^3+2x-2}{x}\,\mathrm{d}x$ c) $\displaystyle\int \frac{6x}{1+3x^2}\,\mathrm{d}x$ d) $\displaystyle\int \frac{\sqrt{x}+1}{x^2}\,\mathrm{d}x$

Problem 8.
Compute the indefinite integrals:

a) $\int (1+e^{2x})\,\mathrm{d}x$ b) $\int e^{1+2x}\,\mathrm{d}x$ c) $\int e^{1-2x}\,\mathrm{d}x$ d) $\int 3^x\,\mathrm{d}x$

Problem 9.
Compute the indefinite integrals:

a) $\displaystyle\int x\sqrt{x^2+1}\,\mathrm{d}x$ b) $\displaystyle\int 9(x+1)^7\,\mathrm{d}x$ c) $\displaystyle\int xe^{-x^2}\,\mathrm{d}x$ d) $\displaystyle\int \frac{x}{1+x^2}\,\mathrm{d}x$ e) $\displaystyle\int \frac{\ln x}{x}\,\mathrm{d}x$

Problem 10.
Compute the indefinite integral:

$$\int \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$$

Problem 11.
Assume that $f(x) \geq 0$ for all x, and that $F(x)$ is a function such that $\int f(x)\,\mathrm{d}x = F(x) + \mathcal{C}$. Is $F(x)$ an increasing function? Explain why/why not.

Problem 12.
We consider the function defined by

$$f(x) = \frac{e^{1-\sqrt{x}}}{\sqrt{x}}, \quad x > 0$$

a. Compute $f'(x)$.

b. Show that f is decreasing in the domain of definition $D_f = (0, \infty)$.

c. Compute the limits

$$\lim_{x\to 0^+} f(x) \quad \text{and} \quad \lim_{x\to\infty} f(x)$$

d. Sketch the graph of f, based on what you found in this problem, and mark the region bounded by the graph of f and the x-axis (for $x > 0$).

Problem 13.
Write down a sum (based on at least $n = 10$ subintervals) that approximates the definite integral

$$\int_0^1 (1-x^2)\,\mathrm{d}x$$

and show the definite integral and its approximation as areas in a figure.

Problem 14.
Problems from the textbook: 9.1.1 - 9.1.11

Answers to Key Problems

Problem 1.

a) 12

b) 176

Problem 2.

a) $\frac{1}{3}x^3 + \mathcal{C}$

b) $2x^4 - 4x^3 + \mathcal{C}$

c) $e^x - 3x^2 + \mathcal{C}$

d) $\frac{1}{9}x^3 - \frac{1}{8}x^4 + \mathcal{C}$

Problem 3.

a) $f(x) = 2x$

b) $f(x) = x^2$

c) $f(x) = 2x^3$

d) $f(x) = \ln(x)$

e) $f(x) = \ln(-x)$

f) $f(x) = \ln|x|$

Problem 4.

a) $f(x) = 0$

b) $f(x) = 2$

c) $f(x) = 12x$

d) $f(x) = (1+2x)e^{2x}$

e) $f(x) = 2x$

f) $f(x) = x^2$

g) $f(x) = 2x^3$

h) $f(x) = \left(\frac{1}{2}x - \frac{1}{4}\right)e^{2x}$

Problem 5.

$A = 1,\ B = 4$

Problem 6.

a) $-\frac{1}{2}x^{-2} + \mathcal{C}$

b) $\frac{2}{3}x\sqrt{x} + \mathcal{C}$

c) $\frac{2}{5}x^2\sqrt{x} + \mathcal{C}$

d) $\ln|x| + \mathcal{C}$

e) $-1/x + \mathcal{C}$

f) $\frac{1}{2}x^2 - \frac{1}{2}x^4 + \mathcal{C}$

g) $\frac{1}{2}x^2 - \frac{2}{3}x^3 + \mathcal{C}$

h) $x(x - x^2 + \mathcal{C})$

i) $\frac{1}{3}(x+1)^3 + \mathcal{C}$

j) $\frac{1}{8}(x+1)^8 + \mathcal{C}$

Problem 7.

a) $-1/x - 3x + \mathcal{C}$

b) $\frac{1}{3}x^3 + 2x - 2\ln|x| + \mathcal{C}$

c) $\ln(1+3x^2) + \mathcal{C}$

d) $-2/\sqrt{x} - 1/x + \mathcal{C}$

Problem 8.

a) $x + \frac{1}{2}e^{2x} + \mathcal{C}$

b) $\frac{1}{2}e^{1+2x} + \mathcal{C}$

c) $-\frac{1}{2}e^{1-2x} + \mathcal{C}$

d) $\frac{1}{\ln 3} \cdot 3^x + \mathcal{C}$

Problem 9.

a) $\frac{1}{3}(x^2+1)^{3/2} + \mathcal{C}$

b) $\frac{9}{8}(x+1)^8 + \mathcal{C}$

c) $-\frac{1}{2}e^{-x^2} + \mathcal{C}$

d) $\frac{1}{2}\ln(1+x^2) + \mathcal{C}$

e) $\frac{1}{2}\ln(x)^2 + \mathcal{C}$

Problem 10.

$-2e^{1-\sqrt{x}} + \mathcal{C}$

Problem 11.

Since $F'(x) = f(x)$ and $f(x) \geq 0$, it follows that F is an increasing function.

Problem 12.

a. $f'(x) = \dfrac{e^{1-\sqrt{x}}(-\sqrt{x}-1)}{2x\sqrt{x}}$

b. Since $f'(x) \leq 0$ for $x > 0$, it follows that f is decreasing

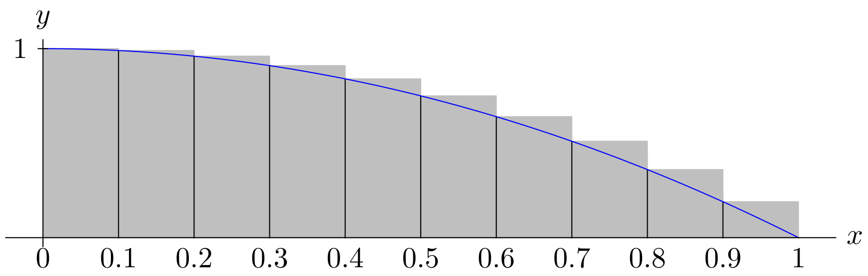
c. $\lim_{x\to 0^+} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = 0$

Problem 13.

We divide [0,1] into $n = 10$ subintervals of length $(1-0)/10 = 1/10$, and the points marking these subintervals are given by $x_i = i/10$ for $i = 0, 1, 2, \ldots, 10$. Hence $x_0 = 0$, $x_1 = 1/10$, $x_2 = 2/10$ and so on. The definite integral is the area under $f(x) = 1 - x^2$ on the interval [0,1]. We can approximate this as the areal of ten rectangles, given by the sum

$$\begin{aligned}\sum_{i=0}^{9} f(x_i)\cdot\Delta x_i &= \sum_{i=0}^{9}\left(1-(i/10)^2\right)\cdot\frac{1}{10} = (1 + (1-1/100) + (1-4/100) + \cdots + (1-81/100))\cdot\frac{1}{10} \\ &= \frac{1}{10}\cdot\left(10 - \frac{0+1+4+\cdots+81}{100}\right) = 0.715\end{aligned}$$

This sum is shown as an area in the figure below. The definite integral is the area under the blue curve, which is a bit smaller than 0.715. The choice $n = 10$ is not important, but the approximation is better the bigger n is.



Problem 14.

See answers in the textbook.