Key Problems

Problem 1.
We consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & 2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 4 \\ 1 & -2 \\ 7 & 1 \end{pmatrix}$$

Compute the following matrices, if possible:

a) $A + B$ b) $2A - 3B$ c) $A - C$ d) AB e) BC f) ABC

g) AC h) A^2 i) BA j) CB k) C^2 l) $C^T A$

Problem 2.
Find A^{-1}, if possible:

a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ b) $A = \begin{pmatrix} 7 & -1 \\ 4 & 2 \end{pmatrix}$ c) $A = \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix}$

d) $A = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ e) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ f) $A = \begin{pmatrix} 7 & 1 & 4 \\ -2 & 1 & -2 \\ 3 & 3 & 0 \end{pmatrix}$

Problem 3.
Determine all values of a such that A is invertible, and compute A^{-1} in these cases:

a) $A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$ b) $A = \begin{pmatrix} 3 & 1 & a \\ 0 & a & 1 \\ 0 & 0 & 2 \end{pmatrix}$ c) $A = \begin{pmatrix} 1 & 1 & a \\ 1 & 3 & 1 \\ a & 1 & 1 \end{pmatrix}$

Problem 4.
We consider the linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} t & 0 & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}$$

a) Solve the system when $t = 2$.

b) Determine the number of solutions for all values of t.

c) Find the inverse matrix A^{-1} when it exists, and use this to solve the linear system in these cases.

Problem 5.
Compute these matrices, and write the answer as simple as possible:

a) $(A+B)^2$ b) $(A^TA)^T$ c) $A(3B-C)+(A-2B)C+2B(C+2A)$

d) $A^{-1}(BA)$ e) $(BAB^{-1})^2 \cdot B^2$ f) $(A-B)(C-A)+(C-B)(A-C)+(C-A)^2$

Problem 6.
Assume that A and B are 3×3-matrices with $|A| = 2$ and $|B| = -5$. Compute:

a) $\det(AB)$ b) $\det(3A)$ c) $\det(-2B^T)$ d) $\det(2A^{-1}B)$

Problem 7.

Solve the matrix equation for X when $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$:

a) $AX = I$ b) $X^2 = A$ c) $AX = XA$

Problem 8.
We consider the linear system $A \cdot \mathbf{x} = \mathbf{b}$ with parameter a, given by

$$A = \begin{pmatrix} 1 & a & 4 \\ 2a & 8 & 12 \\ 5 & 10 & 16 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 11 \\ 40 \\ 51 \end{pmatrix}$$

a. Use Gaussian elimination to solve the linear system when $a = 2$. Mark the pivot positions.

b. Compute $\det(A)$, and determine all values of a such that $\det(A) = 0$.

c. Find A^{-1} when $a = 3$.

d. Show that $A^7 \cdot \mathbf{x} = \mathbf{b}$ has exact one solution for $a = -1$, and express the solution $\mathbf{x}$ in terms of A and $\mathbf{b}$.

Problem 9.
Let A be an $n \times n$ matrix. An elementary row operation $A \to B$ can be realized as a multiplication of an $n \times n$ matrix E from the left, such that $B = E \cdot A$. Then E is called the elementary matrix of the row operation $A \to B$. Find the elementary matrices of the following elementary row operations on 3×3 matrices:

a) Switch the last two rows b) Multiply the second row by -1

c) Add 2 times row one to row two d) Add -2 times row three to row one

Explain why all elementary matrices are invertible, and why a quadratic matrix is invertible if and only if it is a product of elementary matrices.

Problem 10.
Optional: Problems from [Eriksen] (Norwegian textbook)
6.5.1, 6.5.4 - 6.5.6, 6.6.1 - 6.6.6 (textbook) 9.23, 9.25 (workbook)

Answers to Key Problems

Problem 1.

a) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 3 \\ 0 & 3 & 5 \end{pmatrix}$ b) $\begin{pmatrix} 2 & -1 & -4 \\ 1 & 2 & 11 \\ -5 & -4 & -10 \end{pmatrix}$ c) not defined d) $\begin{pmatrix} 2 & 3 & 5 \\ 5 & 10 & 19 \\ 2 & 1 & 1 \end{pmatrix}$

e) $\begin{pmatrix} 15 & 0 \\ -4 & 3 \\ 33 & 4 \end{pmatrix}$ f) $\begin{pmatrix} 44 & 7 \\ 158 & 19 \\ 14 & 7 \end{pmatrix}$ g) $\begin{pmatrix} 11 & 3 \\ 35 & 10 \\ 5 & -5 \end{pmatrix}$ h) $\begin{pmatrix} 2 & 3 & 6 \\ 0 & 7 & 10 \\ 0 & 1 & 4 \end{pmatrix}$

i) $\begin{pmatrix} 0 & 3 & 6 \\ 2 & 0 & 0 \\ 1 & 7 & 13 \end{pmatrix}$ j) not defined k) not defined l) $\begin{pmatrix} -2 & 11 & 14 \\ -1 & 3 & -3 \end{pmatrix}$

Problem 2.

a) $A^{-1} = \frac{1}{3}\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$ b) $A^{-1} = \frac{1}{18}\begin{pmatrix} 2 & 1 \\ -4 & 7 \end{pmatrix}$ c) A^{-1} not defined

d) $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -2 \\ 0 & 2 & -4 \\ 0 & 0 & 2 \end{pmatrix}$ e) $A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$ f) A^{-1} not defined

Problem 3.

a) $A^{-1} = \frac{1}{1-a^2}\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix}$ for $a \neq -1, 1$ b) $A^{-1} = \frac{1}{6a}\begin{pmatrix} 2a & -2 & 1-a^2 \\ 0 & 6 & -3 \\ 0 & 0 & 3a \end{pmatrix}$ for $a \neq 0$

c) $A^{-1} = \frac{1}{(1-a)(1+3a)}\begin{pmatrix} 2 & a-1 & 1-3a \\ a-1 & 1-a^2 & a-1 \\ 1-3a & a-1 & 2 \end{pmatrix}$ for $a \neq -1/3, 1$

Problem 4.

a) $(x,y,z) = (2/3,0,2/3)$

b) Infinitely many solutions for $t = 0$ og $t = 1$, no solutions for $t = -1$, and one solution for $t \neq -1,0,1$

c) $A^{-1} = \frac{1}{t(t^2-1)}\begin{pmatrix} t^2 & 0 & -t \\ 0 & t^2-1 & 0 \\ -t & 0 & t^2 \end{pmatrix}$ for $t \neq -1,0,1$, and $(x,y,z) = \left(\frac{t}{t+1}, 0, \frac{t}{t+1}\right)$ for $t \neq -1,0,1$

Problem 5.

a) $A^2 + AB + BA + B^2$ b) A^TA c) $3AB+4BA$ d) $A^{-1}BA$ e) BA^2B f) 0

Problem 6.

a) -10 b) 54 c) 40 d) -20

Problem 7.

a) $X = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$

b) no solutions

c) $X = s\begin{pmatrix} -4 & 3 \\ 2 & 0 \end{pmatrix} + t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Problem 8.

a) $(7-2y, y, 1)$ where y is free

b) $-32a^2 + 140a - 152$, $a = 2$ or $a = 19/8$

c) $\frac{1}{20}\begin{pmatrix} -8 & 8 & -4 \\ 36 & 4 & -12 \\ -20 & -5 & 10 \end{pmatrix}$

d) $(A^{-1})^7 \cdot \mathbf{b}$

Problem 9.

a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$