Key Problems

Problem 1.
We consider the function $f(x,y) = x^2 - 2x + 4y^2$.

a) Show that the level curve $f(x,y) = c$ is an ellipse when $c > -1$, and determine its center (x_0,y_0) and its half-axes a and b. Use this to sketch the level curves for $c = 0, 1, 2, 3$ in the same coordinate system.

b) Find the tangent to the level curve at $(x,y) = (1,1)$ and at $(x,y) = (2,1/2)$. Show the tangents in the figure.

c) Find $\nabla f(1,1)$ and $\nabla f(2,1/2)$, and show them in the figure. What happens to $f(x,y)$ along the gradients?

d) Does it seem like the function f has a minimum or maximum value? Explain why/why not.

Problem 2.
Find the partial derivatives f'_x and f'_y:

a) $f(x,y) = 2x + 3y$
b) $f(x,y) = x^2 + y^2$
c) $f(x,y) = 4x^2 - 6xy + 9y^2$
d) $f(x,y) = x^2 - 2x + 4y^2$
e) $f(x,y) = x^3 - 3xy + y^3$
f) $f(x,y) = y^2 - x^3 + 3x$
g) $f(x,y) = \sqrt{x^2 + y^2}$
h) $f(x,y) = \ln(x^2y^2 - x^2 - y^2 + 3)$

Problem 3.
Find the Hessian $H(f)$, and compute $H(f)(1,1)$:

a) $f(x,y) = 2x + 3y$
b) $f(x,y) = x^2 + y^2$
c) $f(x,y) = 4x^2 - 6xy + 9y^2$
d) $f(x,y) = x^2 - 2x + 4y^2$
e) $f(x,y) = x^3 - 3xy + y^3$
f) $f(x,y) = y^2 - x^3 + 3x$
g) $f(x,y) = \sqrt{x^2 + y^2}$
h) $f(x,y) = \ln(x^2y^2 - x^2 - y^2 + 3)$

Problem 4.
Find the gradient $\nabla f(1,1)$ of f at $(1,1)$, and use this to find the directional derivative $f'_{\mathbf{a}}(1,1)$ of $f(x,y)$ at $(1,1)$ along the vector $\mathbf{a} = \begin{pmatrix} a_1 & a_2 \end{pmatrix}^T$:

a) $f(x,y) = 2x + 3y$
b) $f(x,y) = x^2 + y^2$
c) $f(x,y) = 4x^2 - 6xy + 9y^2$
d) $f(x,y) = x^2 - 2x + 4y^2$
e) $f(x,y) = x^3 - 3xy + y^3$
f) $f(x,y) = y^2 - x^3 + 3x$
g) $f(x,y) = \sqrt{x^2 + y^2}$
h) $f(x,y) = \ln(x^2y^2 - x^2 - y^2 + 3)$

Problem 5.
Determine the tangent of $f(x,y) = c$ at $(x,y) = (1,1)$:

a) $f(x,y) = 2x + 3y,\ c = 5$
b) $f(x,y) = x^2 + y^2,\ c = 2$
c) $f(x,y) = 4x^2 - 6xy + 9y^2,\ c = 7$
d) $f(x,y) = x^2 - 2x + 4y^2,\ c = 3$
e) $f(x,y) = x^3 - 3xy + y^3,\ c = -1$
f) $f(x,y) = y^2 - x^3 + 3x,\ c = 3$
g) $f(x,y) = \sqrt{x^2 + y^2},\ c = \sqrt{2}$
h) $f(x,y) = \ln(x^2y^2 - x^2 - y^2 + 3),\ c = \ln 2$

Problem 6.
Show that the gradient $\nabla f(a,b)$ is normal to the tangent of the level curve $f(x,y) = c$ at the point (a,b), and that f increases when we move along the direction of the gradient.

Problem 7.
We consider the level curve $f(x,y) = c$ of the function $f(x,y) = x^2 + 4x + y^2 - 2y$. What kind of curve is this? Describe the gradient of f in a point on the level curve geometrically.

Problem 8.
Problem 7.1.1 - 7.1.4, 7.2.1 - 7.2.2, 7.3.1 - 7.3.2 (norwegian textbook, optional)

Answers to Key Problems

Problem 1.
a) Ellipses with center $(1,0)$ and with half-axes $a = \sqrt{c+1}$ and $b = \sqrt{c+1}/2$.
b) The tangents have equations $y = 1$ and $y = -x/2 + 3/2$.
c) $\nabla f(1,1) = \begin{pmatrix} 0 & 8 \end{pmatrix}^T$, $\nabla f(2,1/2) = \begin{pmatrix} 2 & 4 \end{pmatrix}^T$, and the function increases along the gradient.
d) No maximum value (the half-axes increase when c increases). The minimum value is $f(1,0) = -1$.

Problem 2.
a) $f'_x = 2,\ f'_y = 3$
b) $f'_x = 2x,\ f'_y = 2y$
c) $f'_x = 8x - 6y,\ f'_y = -6x + 18y$
d) $f'_x = 2x - 2,\ f'_y = 8y$
e) $f'_x = 3x^2 - 3y,\ f'_y = -3x + 3y^2$
f) $f'_x = -3x^2 + 3,\ f'_y = 2y$
g) $f'_x = \dfrac{x}{\sqrt{x^2+y^2}},\ f'_y = \dfrac{y}{\sqrt{x^2+y^2}}$
h) $f'_x = \dfrac{2x(y^2-1)}{x^2y^2 - x^2 - y^2 + 3},\ f'_y = \dfrac{2y(x^2-1)}{x^2y^2 - x^2 - y^2 + 3}$

Problem 3.
a) $H(f) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ H(f)(1,1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
b) $H(f) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},\ H(f)(1,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
c) $H(f) = \begin{pmatrix} 8 & -6 \\ -6 & 18 \end{pmatrix},\ H(f)(1,1) = \begin{pmatrix} 8 & -6 \\ -6 & 18 \end{pmatrix}$
d) $H(f) = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix},\ H(f)(1,1) = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}$
e) $H(f) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix},\ H(f)(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$
f) $H(f) = \begin{pmatrix} -6x & 0 \\ 0 & 2 \end{pmatrix},\ H(f)(1,1) = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix}$
g) $H(f) = (x^2+y^2)^{-3/2} \cdot \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix},\quad H(f)(1,1) = \begin{pmatrix} \sqrt{2}/4 & -\sqrt{2}/4 \\ -\sqrt{2}/4 & \sqrt{2}/4 \end{pmatrix}$
h) $H(f) = (x^2y^2 - x^2 - y^2 + 3)^{-2} \cdot \begin{pmatrix} 2(y^2-1)(-x^2y^2 + x^2 - y^2 + 3) & 8xy \\ 8xy & 2(x^2-1)(-x^2y^2 - x^2 + y^2 + 3) \end{pmatrix},\quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

Problem 4.

a) $\nabla f(1{,}1) = \begin{pmatrix}2 & 3\end{pmatrix}^T$, $f'_{\mathbf{a}}(1{,}1) = 2a_1 + 3a_2$

b) $\nabla f(1{,}1) = \begin{pmatrix}2 & 2\end{pmatrix}^T$, $f'_{\mathbf{a}}(1{,}1) = 2a_1 + 2a_2$

c) $\nabla f(1{,}1) = \begin{pmatrix}2 & 12\end{pmatrix}^T$, $f'_{\mathbf{a}}(1{,}1) = 2a_1 + 12a_2$

d) $\nabla f(1{,}1) = \begin{pmatrix}0 & 8\end{pmatrix}^T$, $f'_{\mathbf{a}}(1{,}1) = 8a_2$

e) $\nabla f(1{,}1) = \begin{pmatrix}0 & 0\end{pmatrix}^T$, $f'_{\mathbf{a}}(1{,}1) = 0$

f) $\nabla f(1{,}1) = \begin{pmatrix}0 & 2\end{pmatrix}^T$, $f'_{\mathbf{a}}(1{,}1) = 2a_2$

g) $\nabla f(1{,}1) = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}^T$, $f'_{\mathbf{a}}(1{,}1) = (a_1{+}a_2)/\sqrt{2}$

h) $\nabla f(1{,}1) = \begin{pmatrix}0 & 0\end{pmatrix}^T$, $f'_{\mathbf{a}}(1{,}1) = 0$

Problem 5.

a) $y = -2x/3 + 5/3$

b) $y = -x + 2$

c) $y = -x/6 + 7/6$

d) $y = 1$

e) No tangent

f) $y = 1$

g) $y = -x + 2$

h) No tangent

Problem 7.

The curve is a circle with center $(-2{,}1)$ and radius $\sqrt{c+5}$. The gradient points away from the center of the circle.