FORELESNING 10

ELE 3719

BI

EIVIND ERIKSEN

FEB 14 2013

MATEMATIKK

PLAN:

1. Simultent fordelte stokastiske variable
2. Betingede fordelinger

Lærebok:

[R] 2.5 (ikke 2.5.4)
3.1 – 3.3

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1. Simultent fordelte kontinuerlige variable

X, Y simultat ford. kont. } sannsynlighetstetthet $f(x,y)$ slik at

$$p(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

Kumulativ sannsynlighetsfordeling

$$F(a,b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x,y)\, dy\, dx$$

$$\frac{\partial^2 F(a,b)}{\partial a\, \partial b} = f(a,b)$$

Forventningsverdier:

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) \cdot f(x,y)\, dy\, dx$$

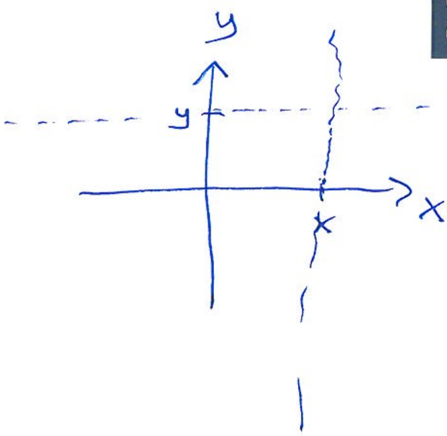
Som i det diskrete tilfellet:

$$Var(X) = E(X^2) - E(X)^2$$
$$Var(Y) = E(Y^2) - E(Y)^2$$
$$Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$$

Regneregler også som i det diskrete tilfellet.

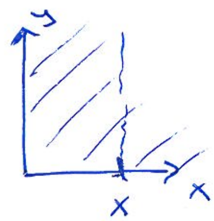
Sannsynlighetstetthet til X/Y:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$



$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Eks: $f(x,y) = \begin{cases} e^{-x-y} & , x,y \geq 0 \\ 0 & , \text{ellers} \end{cases}$



$$f_X(x) = \int_{0}^{\infty} f(x,y)\,dy = \int_0^{\infty} e^{-x-y}\,dy$$

$$= e^{-x}\left[-e^{-y}\right]_0^{\infty} = e^{-x}\Big(-\underset{0}{\underbrace{e^{-\infty}}} + \underset{1}{\underbrace{e^{0}}}\Big) = \underline{\underline{e^{-x}}}$$

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$\leftarrow$ $X \sim Exp(\lambda = 1)$
$E(X) = 1/\lambda$

$$E[\overset{g(x,y)}{X}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^{\infty} \underset{u \cdot v'}{x e^{-x}}\,dx$$

$$= \underset{u \cdot v}{x \cdot (-e^{-x})}\Big|_0^{\infty} - \int_0^{\infty} \underset{u'v}{1 \cdot (-e^{-x})}\,dx$$

$$= -xe^{-x}\Big|_{x=\infty} + 0 + \left[-e^{-x}\right]_0^{\infty}$$

$$= -0 + 0 - e^{-\infty} + e^{0} = \underline{\underline{1}}$$

Uavhengighet:

X og Y uavhengige variable $\Longleftrightarrow$ $f(x,y) = f_X(x) \cdot f_Y(y)$

Nyttig ifm uavhengighet:

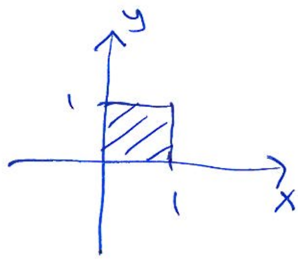
X og Y uavhengige $\Longrightarrow$ $E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)]$

—— " —— $\Longrightarrow$ $Cov(X,Y) = E(XY) - E(X)E(Y) = 0$

Merk: Selv om $Cov(X,Y) = 0$, så er ikke X og Y nødvendigvis uavhengige.

Eks:

$$f(x,y) = \begin{cases} \dfrac{2x+e^y}{e}, & 0 \le x,y \le 1 \\ 0, & \text{ellers} \end{cases}$$



$$Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$$

$$E(XY) = \int_0^1 \int_0^1 xy \cdot \frac{2x+e^y}{e} \, dx\,dy = \frac{1}{e} \iint \underbrace{2x^2y + xye^y}_{} \, dx\,dy$$

$$= \frac{1}{e} \int_0^1 \left[y \cdot \frac{2}{3}x^3 + ye^y \cdot \frac{1}{2}x^2 \right]_0^1 dy$$

$$= \frac{1}{e} \int_0^1 \frac{2}{3}y + \frac{1}{2}\underbrace{ye^y}_{u \cdot v'} \, dy$$

$$= \frac{1}{e}\left[\frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{2}} y^2 \right]_0^1 + \frac{1}{e}\left[\underbrace{y \cdot e^y}_{u \cdot v} - \underbrace{e^y}_{\int u'v} \right]_0^1$$

$$= \frac{1}{3e} \cdot 1 + \frac{1}{e}(e - e - 0 + 1) = \frac{1}{3e} + \frac{1 \cdot 3}{e \cdot 3} = \underline{\frac{4}{3e}}$$

$E(X) = \dots$ $\qquad$ $E(Y) = \dots$

(2) Betingede fordelinger

[R] 3.1–3.3

BI

<u>Husk:</u> $p(E|F) = \dfrac{p(E \cap F)}{p(F)}$

E og F er uavhengige $\Rightarrow p(E|F) = p(E)$

<u>Diskret tilfelle:</u> X, Y simultant fordelte diskrete variabler.

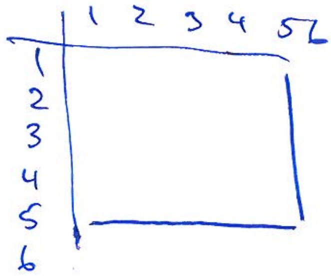
<u>Eks:</u> Vi kaster to terninger

X = max øyne

Y = summen av øynene

$$p(X=5 \mid Y=7) = \frac{p(X=5, Y=7)}{p(Y=7)} = \frac{2/36}{1/6} = \frac{1}{3}$$

$$p(X=5) = \frac{9}{36} = 1/4$$



$$f_{X|Y}(5|7) = 1/3$$

$$f_{X|Y}(x|y) = p(X=x \mid Y=y) = \frac{p(X=x, Y=y)}{p(Y=y)}$$

<u>Generell formel:</u> Betinget fordeling for X gitt Y

$$\boxed{f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}} \quad , \quad f_Y(y) \neq 0$$

$$E[X \mid Y=y] = \sum_x x \cdot f_{X|Y}(x|y)$$

1 eks:

$$E[X \mid Y=7] = \sum_{x=1}^{6} x \cdot f_{X|Y}(x|7)$$

$$= 1 \cdot \overset{=0}{f_{X|Y}(1|7)} + 2 \cdot \overset{=0}{f_{X|Y}(2|7)} + 3 \cdot \overset{=0}{f_{X|Y}(3|7)}$$
$$+ 4 \cdot f_{X|Y}(4|7) + 5 \cdot f_{X|Y}(5|7) + 6 \cdot f_{X|Y}(6|7)$$

$$= 4 \cdot \frac{1}{3} + 5 \cdot \frac{1}{3} + 6 \cdot \frac{1}{3} = \underline{\underline{5}}$$

$f_{X|Y}(x|7)$:

$$p(X=3 \mid Y=7) = \frac{p(X=3, Y=7)}{p(Y=7)} = 0$$
$$p(X=4 \mid Y=7) = 2/36 \,/\, 6/36 = 1/3$$
$$p(X=5 \mid Y=7) = \qquad = 1/3$$
$$p(X=6 \mid Y=7) = \qquad = 1/3$$

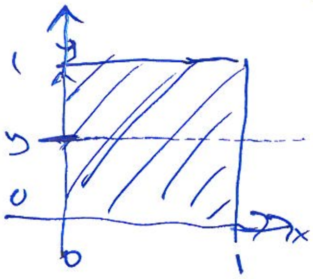
Kontinuerlig tilfelle: X, Y kont., simultant fordelte

Betinget fordeling for X gitt Y:

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}, \quad f_Y(y) \neq 0$$

$$E[X \mid Y=y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\, dx$$

Eks: $f(x,y) = \begin{cases} 6xy(2-x-y) & , 0 \le x,y \le 1 \\ 0 & , \text{ellers} \end{cases}$



$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{6xy(2-x-y)}{4y-3y^2} \quad \begin{pmatrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{pmatrix}$$

$0 \le y \le 1$:
$$f_Y(y) = \int_0^1 f(x,y)\,dx = \int_0^1 6xy(2-x-y)\,dx$$

$$= \int_0^1 12xy - 6x^2y - 6xy^2\,dx$$

$$= \left[y \cdot 6x^2 - y \cdot 2x^3 - y^2 \cdot 3x^2 \right]_{x=0}^{x=1}$$

$$= 6y - 2y - 3y^2 = \underline{4y - 3y^2}$$

$$f_Y(y) = \begin{cases} 4y-3y^2, & 0 \le y \le 1 \\ 0, & \text{ellers} \end{cases}$$

$$E[X|Y=y] = \int_0^1 x \cdot f_{X|Y}(x|y)\,dx = \int_0^1 x \cdot \frac{6xy(2-x-y)}{4y-3y^2}\,dx$$

$$= \frac{1}{4y-3y^2} \cdot \int_0^1 12x^2y - 6x^3y - 6x^2y^2\,dx$$

$$= \frac{1}{4y-3y^2} \cdot \left[y \cdot 4x^3 - y \cdot \frac{6}{4}x^4 - y^2 \cdot 2x^3 \right]_0^1$$

$$= \frac{4y - 1.5y - 2y^2}{4y-3y^2} = \underline{\underline{\frac{2.5y - 2y^2}{4y-3y^2}}}$$