FORELESNING 17

ELE 3719

MATEMATIKK

EIVIND ERIKSEN

MAR 19 2013

PLAN:

① Lineær regresjon
② Kovariansmatriser
③ [Eksamensoppgaver Matriseregning ← Neste gang]
Oppg: Oppgaveark 7

Lærebok:
[MKF] 2.4–2.5

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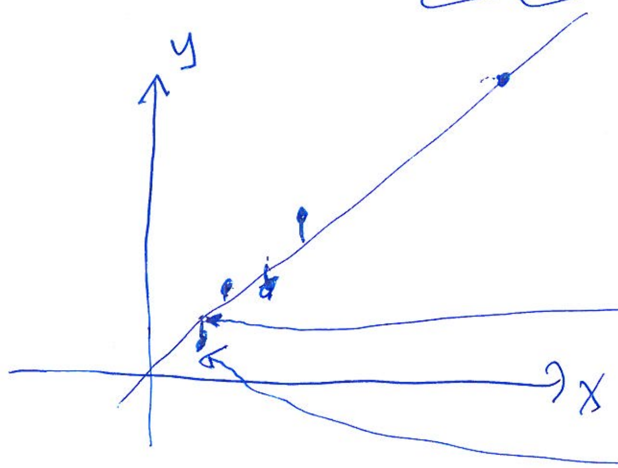
① Lineær regresjon

Eks: Datasett

y	x
1.39	18
1.59	24
9.49	174
2.49	37
3.99	64

x : areal (m^2)
y : pris (m.kr)



Lineær modell:

$$y = \beta_0 + \beta_1 x$$

Datapkt (x_1, y_1)

y-verdi iflg modellen

$$y = \beta_0 + \beta_1 \cdot x_1$$

y_1

Totale feil:

$$\varepsilon = \varepsilon_1^2 + \varepsilon_2^2 + \dots + \varepsilon_5^2$$

Ønsker:
Å finne (β_0, β_1)
slik at ε
bir minst mulig

$$\varepsilon_1 = y - y_1$$
$$= \beta_0 + \beta_1 x_1 - y_1$$

BI

Metode for å finne lineær regresjonslinje:

Datasett:

y	x_1	x_2	x_3	....	x_n
y_1	x_{11}	x_{12}	x_{13}		x_{1n}
y_2	x_{21}	x_{22}	x_{23}		x_{2n}
y_3					
$\vdots$					
y_N	x_{N1}	x_{N2}	x_{N3}		x_{Nn}

N observasjoner
n forklaringsvar.

Lineær modell: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n$

$$(*)\left\{\begin{aligned} y_1 &= \beta_0 + \beta_1 x_{11} + \beta_2 x_{12} + \dots + \beta_n x_{1n} + \varepsilon_1 \\ y_2 &= \beta_0 + \beta_1 x_{21} + \beta_2 x_{22} + \dots + \beta_n x_{2n} + \varepsilon_2 \\ &\;\vdots \\ y_N &= \beta_0 + \beta_1 x_{N1} + \beta_2 x_{N2} + \dots + \beta_n x_{Nn} + \varepsilon_N \end{aligned}\right.$$

Beste tilnærming: $\underline{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$ slik at $\varepsilon = \varepsilon_1^2 + \varepsilon_2^2 + \dots + \varepsilon_N^2$ er minst mulig.

$$(*) \boxed{\underline{y} = X \cdot \underline{\beta} + \underline{\varepsilon}}$$

på matriseform

der $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$, $X = \begin{pmatrix} 1 & x_{11} & x_{12} & \dots & x_{1n} \\ 1 & x_{21} & x_{22} & \dots & x_{2n} \\ 1 & \vdots & & & \vdots \\ 1 & x_{N1} & x_{N2} & & x_{Nn} \end{pmatrix}$, $\underline{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$

$$\underline{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \qquad \underline{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_N \end{pmatrix}$$

Likning: $\underline{y} = X \cdot \underline{\beta} + \underline{\varepsilon}$ $\rightarrow$ Skal velge $\underline{\beta}$ s.a. $\varepsilon = \varepsilon_1^2 + \varepsilon_2^2 + \dots + \varepsilon_N^2$ blir minst mulig

$$\Updownarrow$$

$$\underline{\varepsilon} = \underline{y} - X\underline{\beta}$$

$$\varepsilon = \varepsilon_1^2 + \varepsilon_2^2 + \dots + \varepsilon_N^2 = (\varepsilon_1\ \varepsilon_2\ \varepsilon_3 \dots \varepsilon_N) \cdot \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_N \end{pmatrix} = \underline{\varepsilon}^T \cdot \underline{\varepsilon}$$

$$= (\underline{y} - X\underline{\beta})^T \cdot (\underline{y} - X\underline{\beta})$$

$$= (y^T - \beta^T X^T) \cdot (y - X\beta)$$

$$= \underline{y}^T \cdot \underline{y} - \underline{\beta}^T X^T \cdot \underline{y} - \underline{y}^T X \underline{\beta} + \underline{\beta}^T X^T X \underline{\beta}$$

$$= \underline{y}^T \cdot \underline{y} - (\beta^T X^T \underline{y})^T - \underline{y}^T X \underline{\beta} + \underline{\beta}^T X^T X \underline{\beta}$$

$$= \underline{y}^T \cdot \underline{y} - \underline{y}^T X \underline{\beta} - \underline{y}^T X \underline{\beta} + \underline{\beta}^T X^T X \underline{\beta}$$

$$\boxed{\varepsilon = \underline{\beta}^T (X^T X) \underline{\beta} - 2\underline{y}^T X \cdot \underline{\beta} + \underline{y}^T \underline{y}}$$

$\underline{x}^T A \underline{x} + B\underline{x} + c$

derivert:

$2A \cdot \underline{x} + B^T$

Stasjonære pkt: $\dfrac{\partial \varepsilon}{\partial \underline{\beta}} = \underline{0}$

$$2(X^T X) \cdot \underline{\beta} - (2\underline{y}^T X)^T = \underline{0}$$

$$2X^T X \cdot \underline{\beta} - 2X^T \cdot \underline{y} = \underline{0}$$

$$(X^T X)\,\underline{\beta} = X^T \cdot \underline{y}$$

$X^{-1}(X^T)^{-1} X^T y$

$= X^{-1} \cdot y$

kun hvis $N = n+1$

Anta $|X^T X| \neq 0$ $\longrightarrow$

$$\underline{\beta} = (X^T X)^{-1} \cdot X^T \underline{y}$$

Dette er et minimum fordi $X^T X$ er positiv definit.

(Se oppgave 16, Oppgaveark 7)

2 Kovariansmatriser:

$X_1, X_2, X_3, \dots, X_n$ stokastiske variable $\Bigg\}$ $Y = a_1 X_1 + a_2 X_2 + \dots + a_n X_n$

med $a_1, a_2, \dots, a_n$ er tall

$$E[Y] = E[a_1 X_1 + a_2 X_2 + \dots + a_n X_n]$$
$$= a_1 \cdot E(X_1) + a_2 E(X_2) + \dots + a_n E(X_n)$$

$$\underline{\mu} = \begin{pmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_n) \end{pmatrix} \Rightarrow E(Y) = (a_1, a_2, \dots, a_n) \cdot \underline{\mu}$$

forventningsvektor

$$Var(Y) = Var(a_1 X_1 + a_2 X_2 + \dots + a_n X_n)$$
$$= Cov(a_1 X_1 + a_2 X_2 + \dots + a_n X_n,\ a_1 X_1 + a_2 X_2 + \dots + a_n X_n)$$
$$= a_1^2 Cov(X_1, X_1) + a_1 a_2 Cov(X_1, X_2) + \dots + a_1 a_n \cdot Cov(X_1, X_n)$$
$$+ a_2 a_1 Cov(X_2, X_1) + a_2^2 Cov(X_2, X_2) + \dots$$

$$C = \begin{pmatrix} Cov(X_1,X_1) & Cov(X_1,X_2) & \dots & Cov(X_1,X_n) \\ Cov(X_2,X_1) & \dots & & \\ \vdots & & & \end{pmatrix} \Rightarrow Var(Y) = \underline{a}^T \cdot C \cdot \underline{a}$$

der $\underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

kovariansmatrisen

Kvadratisk form

Husk: $Cov(X_i, X_i) = Var(X_i)$

$Cov(X_i, X_j) = Cov(X_j, X_i)$

(C er symmetrisk)

Oppgave 7, (b)

A $m \times n$-matrise $\Rightarrow$ $B = \underbrace{A^T \cdot A}_{n\times n\text{-matrise}}$

$n\times m$ $\quad$ $m \times n$

* B kvadratisk $(n \times n)$ ✓
* B symmetrisk: $B^T = (A^T A)^T = A^T \cdot A = B$ ✓
* B positiv. semidefinit

B pos. semidef $\Leftrightarrow$ $\lambda_i \geq 0 \ \forall i$

$\Updownarrow$

✓ $\underline{x}^T B \underline{x} \geq 0$ for alle $\underline{x}$

Vi har:

$$\underline{x}^T B \underline{x} = \underline{x}^T A^T A \underline{x} = (A\underline{x})^T \cdot A\underline{x} = \underline{y}^T \cdot \underline{y}$$

$\underline{y} = A\underline{x}$

$$= (y_1\ y_2 \ldots y_n) \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = y_1^2 + y_2^2 + \ldots + y_n^2 \geq 0$$

for alle $\underline{x}$

Derfor er $\underline{x}^T B \underline{x} \geq 0$ for alle $\underline{x}$.

* Hvis A er kvadratisk og invertibel, så er B pos. defn

B pos defn

$\Updownarrow$

$\lambda_1, \lambda_2, \ldots \lambda_n > 0$ ✓

* Vet at B pos. semidefinit, dvs $\lambda_i \geq 0$.
* Hvis én egenverdi $\lambda_i = 0$, så er $\det(B) = \lambda_1 \lambda_2 \ldots \lambda_n = 0$.

Hvis A kvadr + invertibel, så $|A| \neq 0$, og $|B| = |A^T \cdot A|$

$= |A^T| \cdot |A| = |A| \cdot |A| \neq 0$.

Derfor er $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$.

(15) A symm. 2x2

Kar. likn: $\lambda^2 - tr(A)\cdot\lambda + det(A) = 0$

Egenverdier: λ_1, λ_2 slik at $\begin{cases} \lambda_1 + \lambda_2 = tr(A) \\ \lambda_1 \cdot \lambda_2 = det(A) \end{cases}$

a) A pos. defn

$\Updownarrow$

$\lambda_1, \lambda_2 > 0 \iff tr(A) > 0,\ det(A) > 0$

b) A indefinit

$\Updownarrow$

$\lambda_1 > 0,\ \lambda_2 < 0 \iff det(A) < 0$