FORELESNING 23

ELE 3719

BI

EIVIND ERIKSEN   APR 16 2013   MATEMATIKK

PLAN:

1. Eksamen V'12 Opps 5–6
   Eksamen H'11 (konte)
2. Eksamen V'11 Opps 4

$\underbrace{\text{Eksamen V'11}}_{\text{ELE3719}}$

Eldre eks.oppgaver

MET2214

Oppgavesett 10

Euler:

$$F'_y - \frac{d}{dt}(F'_{\dot y}) = 0$$

4) $\min \int_{t_0}^{t_1} F(t, y, \dot y)\, dt$

a) $F = 2ty + 3y\dot y + t\dot y^2$

$y = y(t) \quad \dot y = y'(t) \quad \ddot y = y''(t)$

$F'_y = 2t + 3\dot y$

$F'_{\dot y} = 3y + t \cdot 2\dot y \qquad \frac{d}{dt}(3\underline{y} + t \cdot 2\dot y) = 3\dot y + 1 \cdot 2\dot y + t \cdot 2\ddot y$

$= 5\dot y + 2t\ddot y$

$(2t + 3\dot y) - (5\dot y + 2t\ddot y) = 0$

$$\boxed{2t - 2\dot y - 2t\ddot y = 0}$$

b) $F = \cancel{-}\, e^{\dot{y}-ay}$

$F'_y = -e^{\dot{y}-ay}\cdot(-a) = a e^{\dot{y}-ay}$

$F'_{\dot{y}} = -e^{\dot{y}-ay}\cdot 1 \quad = -e^{\dot{y}-ay}$

$\frac{d}{dt}(F'_{\dot{y}}) = -e^{\dot{y}-ay}\cdot(\ddot{y}-a\dot{y})$

Euler: $a\cdot e^{\dot{y}-ay} + (y''-a\dot{y})\, e^{\dot{y}-ay} = 0$

$a + (y''-ay') = 0$

$y''-ay'+a=0$

c) $F = [(y-\dot{y})^2+y^2]\, e^{-at}$

$F'_y = [2(y-\dot{y})\cdot 1 + 2y]\, e^{-at}$

$F'_{\dot{y}} = [2(y-\dot{y})\cdot(-1)\quad]\, e^{-at}$

$\frac{d}{dt}\left[-2(y-\dot{y})\, e^{-at}\right] = -2(\dot{y}-\ddot{y})\, e^{-at}$

$+ (-2)(y-\dot{y})\cdot e^{-at}\cdot(-a)$

$= e^{-at}\left[-2(\dot{y}-\ddot{y}) + 2a(y-\dot{y})\right]$

Euler: $[2(y-\dot{y})+2y]\cancel{e^{-at}} - [-2(\dot{y}-\ddot{y})+2a(y-\dot{y})]\cancel{e^{-at}} = 0$

$-2\ddot{y} + 2a\dot{y} + (4-2a)y = 0$

① Elo. V'12

5) a) $yy' = 1$, $y(5) = -3$

$y' = \frac{1}{y} \cdot 1$ separabel

$yy' = 1$

$\int y\,dy = \int 1\,dt$

$\frac{1}{2}y^2 = t + C$

$y^2 = 2t + 2C$ $\qquad (-3)^2 = 2 \cdot 5 + 2C$

$y^2 = 2t - 1$ $\qquad 9 = 10 + 2C$

$y = -\sqrt{2t - 1}$ $\qquad 2C = -1$

b) $y' - ty = t$, $y(0) = 2$

Int. faktor:

$u = e^{\int -t\,dt} = e^{-\frac{1}{2}t^2}$

$\left(y \cdot e^{-\frac{1}{2}t^2}\right)' = t \cdot e^{-\frac{1}{2}t^2}$

$y \cdot e^{-\frac{1}{2}t^2} = \int t e^{-\frac{1}{2}t^2}\,dt = -e^{-\frac{1}{2}t^2} + C$

$y = -1 + C \cdot e^{\frac{1}{2}t^2}$

$y' = t + ty = (1+y) \cdot t$

$\frac{1}{1+y} \cdot y' = t$

$\int \frac{1}{1+y}\,dy = \int t\,dt$

$\ln|1+y| = \frac{1}{2}t^2 + C$

$|1+y| = e^{\frac{1}{2}t^2} \cdot e^C$

$1 + y = \pm e^C \, e^{\frac{1}{2}t^2}$

$y = -1 + K \cdot e^{\frac{1}{2}t^2}$

$y(0) = 2$: $\quad 2 = -1 + C \cdot 1 \implies y = -1 + 3e^{\frac{1}{2}t^2}$

$C = 3$

c) $y'' = y + 2$, $y(0) = 0$, $y'(0) = 0$

$y'' - y = 2$

$y = y_h + y_p = \underline{C_1 e^{t} + C_2 e^{-t} - 2}$

$\underline{y_h}$: $y'' - y = 0$

$r^2 - 1 = 0$

$r = \pm 1 \implies y_h = \underline{C_1 \cdot e^{t} + C_2 e^{-t}}$

$\underline{y_p}$: $y'' - y = \underline{\underline{2}}$ Gjetter: $\underline{y = A}$, $y' = 0$, $y'' = 0$

$0 - A = 2$

$A = -2$

$y_p = \underline{-2}$

$y = \underline{C_1 e^{t} + C_2 e^{-t} - 2}$ $y' = C_1 e^{t} - C_2 e^{-t}$

$y(0) = 0$: $0 = C_1 + C_2 - 2 \implies C_1 + C_2 = 2$

$y'(0) = 0$: $0 = C_1 - C_2 \implies \underline{C_1 = C_2}$

$C_1 = C_2 = 1$

$$y = \underline{\underline{e^{t} + e^{-t} - 2}}$$

6) $F = \left(3\dot{y}^2 + (y-\dot{y})^2\right) e^{-\rho t}$ $(\rho > 0)$

a) $F'_y = 2(y-\dot{y})e^{-\rho t}$

$F'_{\dot{y}} = [6\dot{y} - 2(y-\dot{y})]e^{-\rho t}$

$F''_{yy} = 2e^{-\rho t}$

$F''_{y\dot{y}} = -2e^{-\rho t}$

$F''_{\dot{y}y} = -2e^{-\rho t}$

$F''_{\dot{y}\dot{y}} = (6+2)e^{-\rho t}$
$= \boxed{8e^{-\rho t}}$

$F''_{yy} \cdot F''_{\dot{y}\dot{y}} - (F''_{y\dot{y}})^2 = 16(e^{-\rho t})^2 - 4(e^{-\rho t})^2$
$= 12e^{-2\rho t} > 0$ for alle $y, \dot{y}$

$F''_{yy} = 2e^{-\rho t} > 0$, $F''_{\dot{y}\dot{y}} = 8e^{-\rho t} > 0$ — " —

$\Downarrow$

F er konveks i $(y, \dot{y})$

b) $\rho = 0$:

$F'_y = 2(y-\dot{y}) = 2y - 2\dot{y}$

$F'_{\dot{y}} = 6\dot{y} - 2(y-\dot{y})$
$= 8\dot{y} - 2y$

$\frac{d}{dt}(8\dot{y} - 2y)$
$= 8\ddot{y} - 2\dot{y}$

Euler: $(2y - 2\dot{y}) - (8\ddot{y} - 2\dot{y}) = 0$

$\boxed{-8\ddot{y} + 2y = 0}$

$-8r^2 + 2 = 0$

$r^2 = 1/4$ $r = \pm 1/2$

$y = C_1 \cdot e^{\frac{1}{2}t} + C_2 e^{-\frac{1}{2}t}$

$$y = c_1 e^{\frac{1}{2}t} + c_2 \cdot e^{-\frac{1}{2}t}$$

$y(0) = 0$:

$y(2) = \dfrac{e^2 - e^{-2}}{2}$

$$0 = c_1 \cdot e^0 + c_2 \cdot e^0 = c_1 + c_2$$

$$\Rightarrow c_2 = -c_1$$

$$\frac{e^2 - e^{-2}}{2} = c_1 \cdot e^1 - c_1 e^{-1}$$

$$\frac{e^2 - e^{-2}}{2} = c_1 \cdot (e - e^{-1})$$

$$c_1 = \frac{(e^2 - e^{-2}) \cdot e^2}{2 \cdot (e - e^{-1}) \cdot e^2}$$

$$= \frac{\cancel{e^4 - 1}\;\; e^2 + 1}{2e(\cancel{e^2 - 1})} = \underline{\underline{\frac{1}{2}\,\frac{e^2+1}{e}}}$$

$$\underline{\underline{y = \frac{1}{2}\,\frac{e^2+1}{e}\left(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}\right)}}$$

gir minimum siden F konveks.

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Eb. V'11

4) a) $\underline{\min} \int_0^4 \underbrace{K\dot{y}^2 \cdot e^{-rt}} dt$, $y(0)=0$, $y(\cancel{4})=30$

Må sjekke om $F = K \cdot \dot{y}^2 \cdot e^{-rt}$ er <u>konveks</u>.

$F'_y = 0$

$F'_{\dot{y}} = \underline{2K\dot{y} \cdot e^{-rt}}$

$F''_{yy} = 0$ $\quad$ $F''_{yy'} = 0$

$F''_{y\dot{y}} = 0$ $\quad$ $F''_{\dot{y}\dot{y}} = 2K \cdot e^{-rt}$

gjelder for alle $(y, \dot{y})$ $\left\{ \begin{array}{l} \det = 0 \cdot 2Ke^{-rt} - 0^2 = 0 \geq 0 \\ F''_{yy} = 0, \ F''_{\dot{y}\dot{y}} = \underline{2Ke^{-rt}} \geq 0 \\ \text{(siden K positiv)} \end{array} \right.$

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F konveks

$\Downarrow$

<u>y^* er minimum</u>

b) <u>Euler:</u> $F'_y = 0$

$F'_{\dot{y}} = 2K\dot{y}e^{-rt}$ $\quad$ $\frac{d}{dt}\left(2K\dot{y}e^{-rt}\right)$

$= 2K\ddot{y}e^{-rt} + 2K\dot{y} \cdot e^{-rt}(-r)$

$= e^{-rt}(2K\ddot{y} - 2Kr\dot{y})$

$\cancel{0} - \cancel{e^{-rt}}(\cancel{2K}\ddot{y} - \cancel{2K}r\dot{y}) = 0$

$\boxed{\ddot{y} - r\dot{y} = 0}$

Euler: $\ddot{y} - r\dot{y} = 0$

$\ddot{y} - \rho \cdot \dot{y} = 0$

$r^2 - \rho r = 0$

$r = 0 \;,\; r = \rho$

r = diskonterings-raten
Vi haller disk.r. for ρ
(for å unngå sammen-blanding)

Hvis $\rho \neq 0$: $y = C_1 \cdot e^{0t} + C_2 \cdot e^{\rho t}$

$= C_1 + C_2 e^{\rho t}$

$= C_1 + C_2 e^{rt}$

$y(0) = 0$: $0 = C_1 + C_2 \cdot 1 \Rightarrow C_1 + C_2 = 0$

$y(4) = 30$: $30 = C_1 + C_2 \cdot e^{4r} \Rightarrow C_1 + C_2 \cdot e^{4r} = 30$

$C_1 + C_2 = 0$
$C_1 + C_2 \cdot e^{4r} = 30$

$\Rightarrow C_2 = -C_1$

$C_1 - C_1 \cdot e^{4r} = 30$

$C_1 \cdot (1 - e^{4r}) = 30$

$C_1 = \dfrac{30}{1 - e^{4r}}$

$$y = \frac{30}{1 - e^{4r}} \cdot \left(1 - e^{rt}\right)$$

$$y^* = 30 \cdot \frac{1 - e^{rt}}{1 - e^{4r}}$$ er minimum

Minste verdi:

$$y = 30\,\frac{1-e^{rt}}{1-e^{4r}} \qquad \dot{y} = \frac{30}{1-e^{4r}}\cdot(-r)e^{rt}$$

$$\dot{y}^2 = \frac{900\,r^2}{(1-e^{4r})^2}\,e^{2rt}$$

$$\int_0^4 K\dot{y}^2 e^{-rt}\,dt = K\cdot\frac{900\,r^2}{(1-e^{4r})^2}\int_0^4 e^{rt}\,dt$$

$$= 10.000\cdot\frac{900\cdot 0.08^2}{(1-\cancel{e}^{0.32})^2}\left[\frac{e^{0.08t}}{0.08}\right]_0^4$$

$$= 10000\cdot\frac{900\cdot 0.08^2}{(1-e^{0.32})^2\cdot 0.08}\cdot(e^{0.32}-1)$$

$$= \frac{10000\cdot 900\cdot 0.08}{e^{0.32}-1} \approx \underline{\underline{1.909.167}}$$