FORELESNING 8

ELE 3719

BI

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FEB 07 2013

MATEMATIKK

PLAN:

① Grenseresultater

② Simultane fordelinger

Lærebok:

[R] 2.5, 2.8

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① Grenseresultater:

Markov's ulikhet: $X \geq 0$

$$P(X \geq a) \leq \frac{E(X)}{a} \quad \text{for } a > 0$$

Bevis: (kont. tilfelle)

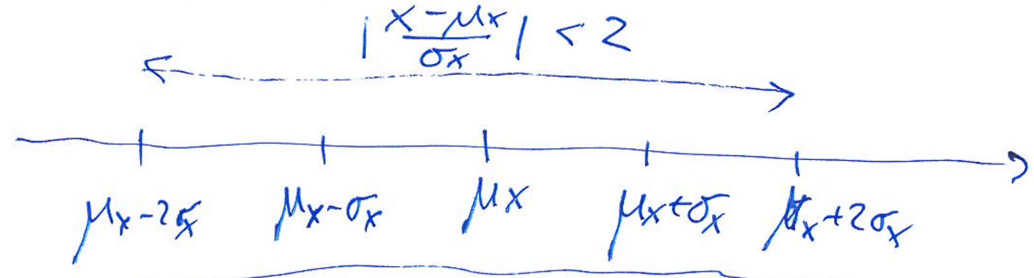
$$E(X) = \int_0^\infty x f(x)\,dx = \underbrace{\int_0^a x f(x)\,dx}_{\geq 0} + \int_a^\infty x f(x)\,dx \geq \int_a^\infty x f(x)\,dx$$

$$\geq \int_a^\infty a f(x)\,dx = a \int_a^\infty f(x)\,dx = a \cdot P(X \geq a)$$

$$E(X) \geq a \cdot P(X \geq a) \Rightarrow \boxed{\frac{E(X)}{a} \geq P(X \geq a)} \quad \square$$

Chebyshev's ulikhet: X stokastisk variabel, $k>0$

$$P\left(\left|\frac{X-\mu_X}{\sigma_X}\right| > k\right) \leq 1/k^2$$



$$P \geq 75\%$$

$$\Rightarrow \qquad P \leq 25\%$$

Bevis:

Markov's ulikhet: $Y = (X-\mu_X)^2$, $a = k^2\sigma_X^2$

$$P\left(\left|\frac{X-\mu_X}{\sigma_X}\right| > k\right) = P\Big(\underbrace{(X-\mu_X)^2}_{\geq 0} > k^2\sigma_X^2\Big) \leq \frac{E\left[(X-\mu_X)^2\right]}{k^2\sigma_X^2} = \frac{\sigma_X^2}{k^2\sigma_X^2} = \frac{1}{k^2} \quad \square$$

$$P\left(\left|\frac{X-\mu_X}{\sigma_X}\right| > k\right) \leq \frac{1}{k^2}$$

Store talls lov (sterk form):

$X_1, X_2, X_3, \ldots$ uavhengige stokastiske variable med samme fordeling $\begin{pmatrix} \mu = \mu_{X_1} = \mu_{X_2} = \ldots \\ \sigma = \sigma_{X_1} = \sigma_{X_2} = \ldots \end{pmatrix}$

$$\frac{X_1 + X_2 + \ldots + X_n}{n} \to \mu, \quad \text{når } n \to \infty$$

DVS:

$$P\left(\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| > \varepsilon\right) \to 0 \quad \text{når } n \to \infty \quad \text{for } \varepsilon > 0 \text{ liten}$$

Bevis: $Z_n = \dfrac{X_1 + X_2 + \dots + X_n}{n}$

$\mu(Z_n) = \dfrac{E(X_1) + \dots + E(X_n)}{n}$

$= \dfrac{n \cdot \mu}{n} = \mu$

$\sigma(Z_n) = \dfrac{\sigma}{\sqrt{n}}$

$$P\left(|Z_n - \mu| \geq \varepsilon\right) = P\left(\left|\frac{Z_n - \mu}{\sigma(Z_n)}\right| > \frac{\varepsilon}{\sigma(Z_n)}\right)$$

$$\leq \frac{1}{(\varepsilon/\sigma(Z_n))^2} = \frac{\sigma(Z_n)^2}{\varepsilon^2}$$

$$= \frac{\sigma^2/n}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2 \cdot n} \to 0 \quad \text{når } n \to \infty.$$

$\blacksquare$

Merknad: Hvorfor er $\sigma(Z_n) = \sigma/\sqrt{n}$?

$$\mathrm{Var}(Z_n) = \mathrm{Var}\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right) = \frac{1}{n^2}\mathrm{Var}(X_1 + X_2 + \dots + X_n)$$

bruker uavhengighet

$$= \frac{1}{n^2}\left[\mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \dots + \mathrm{Var}(X_n)\right] = \frac{n \cdot \sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$$\sigma_{Z_n} = \sqrt{\mathrm{Var}(Z_n)} = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$$

$$\mathrm{Var}(X_1 + X_2) = E\left[(X_1 + X_2)^2\right] - E\left[X_1 + X_2\right]^2$$

$$= E\left[X_1^2 + 2X_1X_2 + X_2^2\right] - \left[E(X_1) + E(X_2)\right]^2$$

$$= E(X_1^2) + 2E(X_1X_2) + E(X_2^2) - E(X_1)^2 - 2E(X_1)E(X_2) - E(X_2)^2$$

Her må vi bruke kovarians for simultane fordelinger

$$= E(X_1^2) - E(X_1)^2 + E(X_2^2) - E(X_2)^2 + 2E(X_1X_2) - 2E(X_1)E(X_2)$$

$$= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2 \cdot \underbrace{\mathrm{Cov}(X_1, X_2)}_{=\,0}$$

pga uavhengighet

$$= \mathrm{Var}(X_1) + \mathrm{Var}(X_2)$$

2 Simultene fordelinger

X, Y stokastiske variable

i) Diskret tilfelle: X og Y er diskrete variable

$$f(x,y) := p(X=x, Y=y) = p(X=x \text{ og } Y=y)$$

X og Y er uavhengige huis $p(X=x, Y=y) = p(X=x) \cdot p(Y=y)$

dvs

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

Defn: X og Y er uavhengige huis $f(x,y) = f_X(x) \cdot f_Y(y)$ for alle (x,y).

$$f(x,y) = p(X=x, Y=y) \qquad f_X(x) = p(X=x) \qquad f_Y(y) = p(Y=y)$$

Eks: Vi kaster ~~en~~ terning to ganger.

X = antall øyne på første terning

Y = summen av antall øyne på begge terninger

$$\left.\begin{array}{l} p(X=3, Y=7) = 1/36 \\ p(X=3) = 1/6 \qquad p(Y=7) = 6/36 \end{array}\right\} \quad f(3,7) = f_X(3) \cdot f_Y(7)$$

$$\left.\begin{array}{l} p(X=3, Y=11) = 0 \\ p(X=3) = 1/6 \qquad p(Y=11) = 2/36 \end{array}\right\} \quad \begin{array}{l} f(3,11) \neq f_X(3) \cdot f_Y(11) \\ X, Y \text{ ikke uavh.} \end{array}$$

X\Y	2	3	4	5	6	7	8	9	10	11	12
1											
2			•								
3											
4											
5											
6											

$$f(2,4) = P(X=2, Y=4) = 1/36$$

<u>Eks:</u>

	Y=1	Y=2	Y=3
X=1	0.20	0.15	0.20
X=2	0.10	0.15	0.20

$$\left(\begin{array}{l} f(1,1) = 0.20 \ \ldots \\ f(2,1) = 0.10 \ \ldots \end{array} \right)$$

<u>Krav til $f(x,y)$:</u>

i) $f(x,y) \geq 0$ for alle x,y

ii) $\sum_{x,y} f(x,y) = 1$

$$E(X) = \sum_{x,y} x \cdot f(x,y) = 1 \cdot \underbrace{(0.20 + 0.15 + 0.20)}_{P(X=1)} + 2 \cdot \underbrace{(0.10 + 0.15 + 0.20)}_{P(X=2)}$$

$$= 1 \cdot f_X(1) + 2 \cdot f_X(2) = \underline{\underline{1.45}}$$

$$E(Y) = \sum_{x,y} y \cdot f(x,y) = 1 \cdot (0.20 + 0.10) + 2 \cdot (0.15 + 0.15) + 3 \cdot (0.20 + 0.20)$$

$$= 1 \cdot f_Y(1) + 2 \cdot f_Y(2) + 3 \cdot f_Y(3) = \underline{\underline{2.10}}$$

X, Y simultant fordelt med tetthet $f(x,y)$, så har vi:

$$f_X(x) = \sum_y f(x,y) = f(x,y_1) + f(x,y_2) + \dots + f(x,y_m)$$
$$f_Y(y) = \sum_x f(x,y) = f(x_1,y) + f(x_2,y) + \dots + f(x_n,y)$$

Forventningsverdiene er gitt ved:

$$\boxed{\begin{aligned} E(X) &= \sum_{x,y} x \cdot f(x,y) = \sum_x x \cdot f_X(x) \\ E(Y) &= \sum_{x,y} y \cdot f(x,y) = \sum_y y \cdot f_Y(y) \\ E[g(X,Y)] &= \sum_{x,y} g(x,y) \cdot f(x,y) \end{aligned}}$$

Eks:

x \ Y	1	2	3
1	0.20	0.15	0.20
2	0.10	0.15	0.20

$$E(XY) = \sum_{x,y} xy \cdot f(x,y)$$
$$= 1 \cdot 1 \cdot 0.20 + 1 \cdot 2 \cdot 0.15 + 1 \cdot 3 \cdot 0.20$$
$$+ 2 \cdot 1 \cdot 0.10 + 2 \cdot 2 \cdot 0.15 + 2 \cdot 3 \cdot 0.20 = \underline{3.1}$$

$$\mathrm{Cov}(X,Y) = E[(X-\mu_X)\cdot(Y-\mu_Y)]$$
$$= E[XY - \mu_X Y - X \cdot \mu_Y + \mu_X \mu_Y]$$
$$= E(XY) - \cancel{\mu_X \cdot E(Y)} - \mu_Y \cdot E(X) + \cancel{\mu_X \mu_Y}$$
$$= E(XY) - \mu_X \mu_Y$$

$$\boxed{\mathrm{Cov}(X,Y) = E(XY) - E(X) \cdot E(Y)}$$

$\mathrm{Cov}(X,X) = \mathrm{Var}(X)$

I eks:

$\mathrm{Cov}(X,Y) = E(XY) - E(X) \cdot E(Y)$

$= 3.1 - 1.45 \cdot 2.10$

$= \underline{\underline{0.055}}$

Resultat:

Hvis X og Y er uavhengige, så har vi:

$$E[g(X)\cdot h(Y)] = E[g(X)]\cdot E[h(Y)].$$

Spesialtilfelle:

X, Y uavhengige $\Rightarrow E[X\cdot Y] = E(X)\cdot E(Y)$

Dvs: Hvis X og Y er uavhengige, så er $\operatorname{Cov}(X,Y) = 0$.

$$\operatorname{Cov}(X,Y) = E(XY) - E(X)\cdot E(Y)$$

Merk: Det kan hende at $\operatorname{Cov}(X,Y) = 0$ selv om X og Y ikke er uavhengige.

Noen ting er akkurat som i tilfellet med én variabel

$$\operatorname{Var}(X) = E[(X-\mu_X)^2] = E(X^2) - E(X)^2$$
$$\operatorname{Var}(Y) = E[(Y-\mu_Y)^2] = E(Y^2) - E(Y)^2$$

Regneregler:

$$E[aX + bY + c] = aE(X) + bE(Y) + c \qquad (a, b, c \text{ konstant})$$

$\operatorname{Var}(X+Y) = ?$

Svar: Se under store talls lov

$$\operatorname{Var}(X+Y) = \operatorname{Var}(X) + \operatorname{Var}(Y) + 2\operatorname{Cov}(X,Y)$$