ELE 3719, 9. forelesning, 7. okt. 2025, Runar Ile

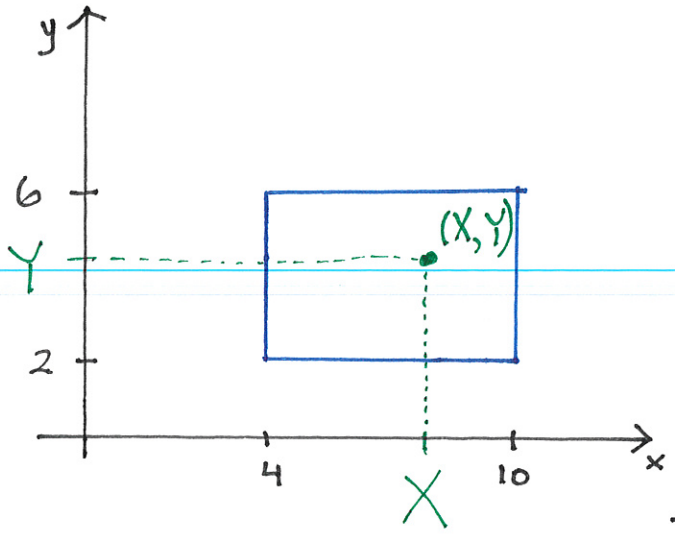
1. Repetisjon
2. Kummulativ fordelingsfunksjon (to var.) (3.3)
3. Marginalene
4. Forventning, varians, kovarians

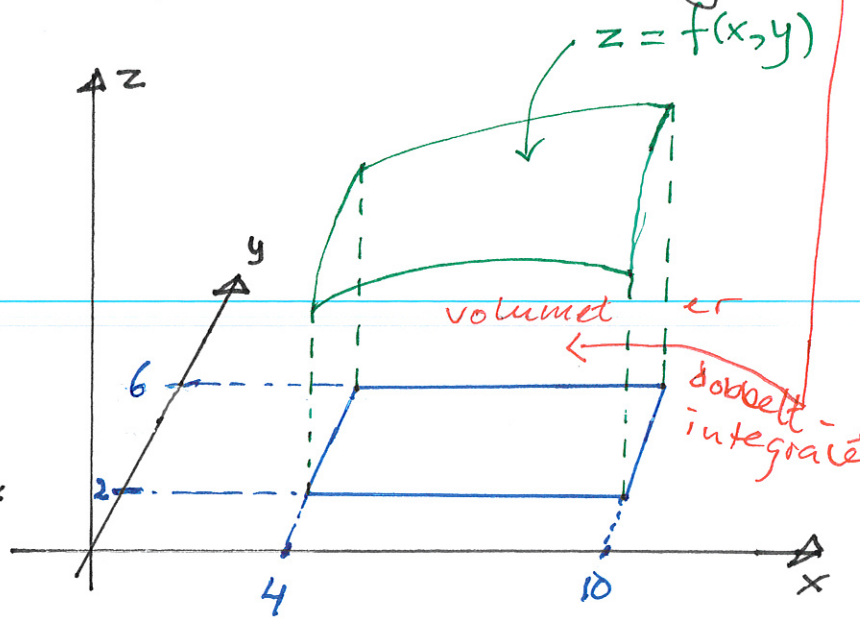
1. Repetisjon

Vi har ett forsøk og stok. variabler X og Y. Da finnes det en simultan tetthetsfunksjon $f(x,y)$ slik at dobbeltintegraler gir sannsynligheter,

f.eks. $P(4 \le X \le 10,\ 2 \le Y \le 6) = \int_2^6 \int_4^{10} f(x,y)\,dx\,dy$

= sanns. for at punktet (X,Y) havner i rektanglet





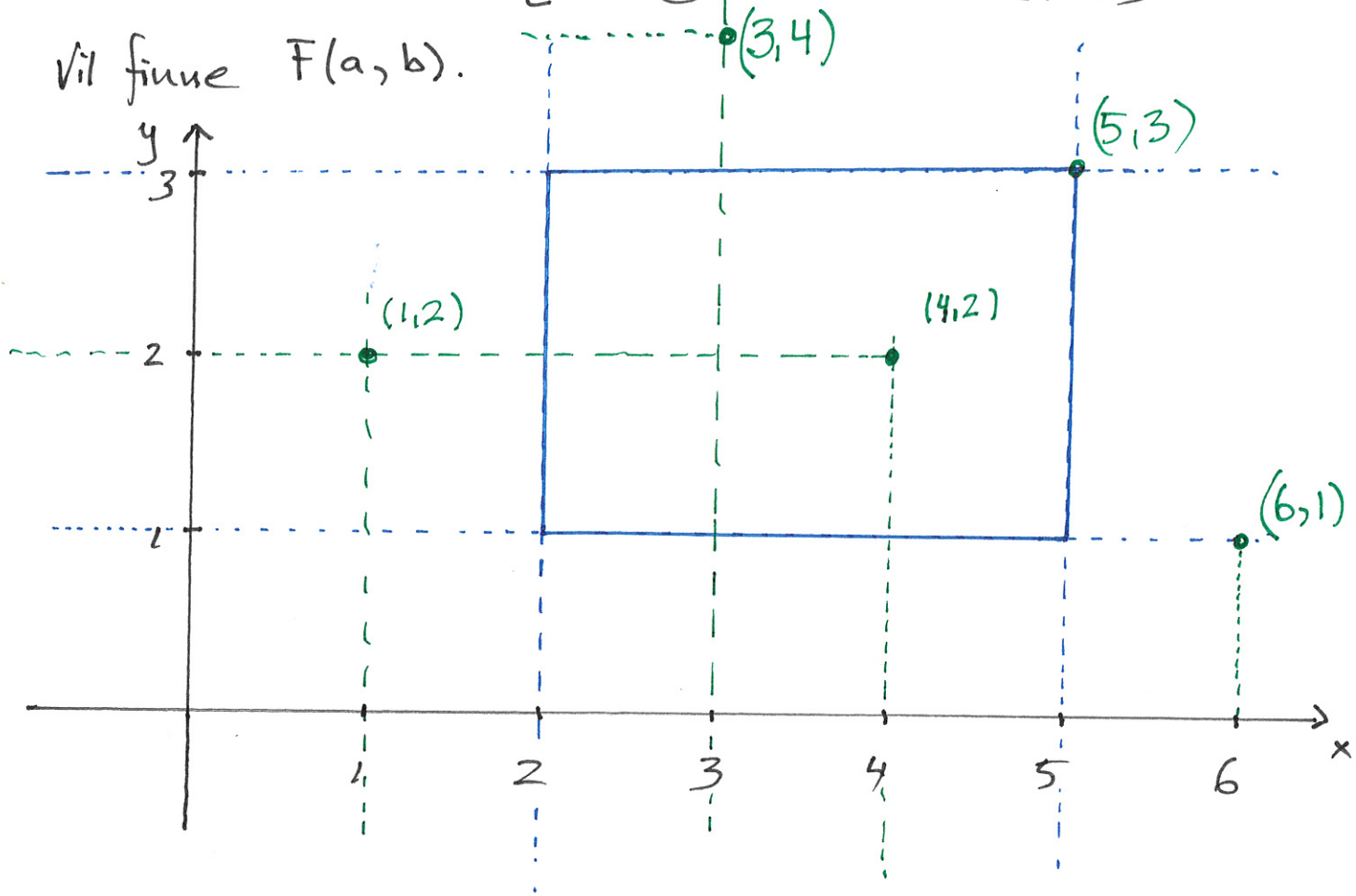
2. Kummulativ fordelingsfunksjon

$$F(a,b) = P(X \le a,\ Y \le b)$$

= sanns. for at X er mindre eller lik a og at Y er mindre eller lik b.

Eks (3.8) $f(x,y) = \begin{cases} \frac{1}{91} xy^2 & \text{for } 2 \le x \le 5 \text{ og } 1 \le y \le 3 \\ 0 & \text{ellers} \end{cases}$

Vil finne $F(a,b)$.



$F(4,2) = P(X \le 4, Y \le 2)$ = samms. for at punktet (X,Y) havner i det uendelig store „rektanglet" med øvre høyre hjørne $(4,2)$.

Altså er $F(4,2) = \int_{-\infty}^{2} \int_{-\infty}^{4} f(x,y)\, dx\, dy$

$$= \int_{1}^{2} \int_{2}^{4} \frac{1}{91} \cdot x \cdot y^2 \, dx\, dy$$

Oppg Bestem integralgrensene / sett opp dobbelt-integralet

$$F(1,2) = P(X \le 1, Y \le 2) = 0$$

$$F(3,4) = P(X \le 3, Y \le 4) = \int_1^3 \int_2^3 \frac{1}{91} x y^2 \, dx\,dy$$

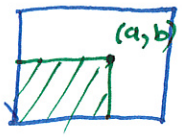
$$F(6,1) = P(X \le 6, Y \le 1) = 0$$

$$F(5,3) = P(X \le 5, Y \le 3) = \int_1^3 \int_2^5 \frac{1}{91} x y^2 \, dx\,dy = 1$$

Start: 15.00

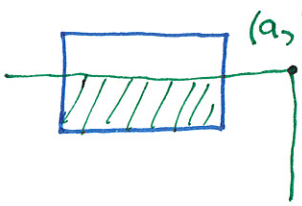
$P(X \le a, Y \le b) = F(a,b) =$

$$\int_1^b \int_2^a \frac{1}{91} x y^2 \, dx\,dy$$



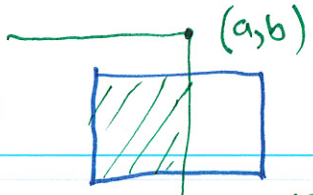
$2 \le a \le 5$ og $1 \le b \le 3$

$$\int_1^b \int_2^5 \frac{1}{91} x y^2 \, dx\,dy$$



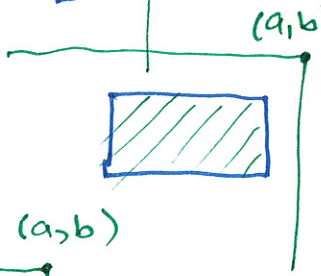
$a \ge 5$ og $1 \le b \le 3$

$$\int_1^3 \int_2^a \frac{1}{91} x y^2 \, dx\,dy$$



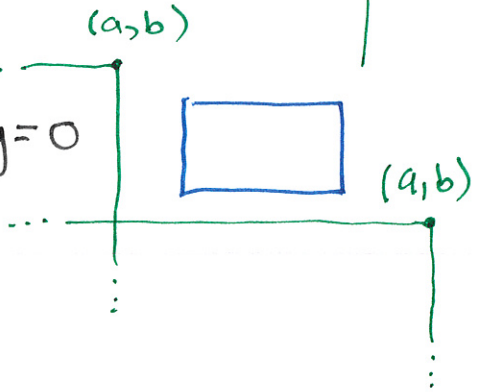
$2 \le a \le 5$ og $3 \le b$

$$\int_1^3 \int_2^5 \frac{1}{91} x y^2 \, dx\,dy = 1$$



$a \ge 5$ og $b \ge 3$

$$\int_{-\infty}^b \int_{-\infty}^a \frac{1}{91} x y^2 \, dx\,dy = 0$$

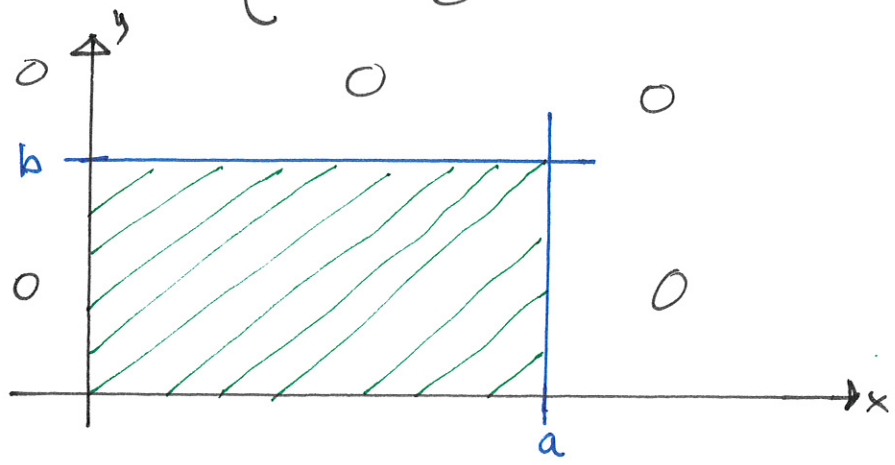


Enten $a \le 2$ eller $b \le 1$

Eks $f(x,y) = \begin{cases} 0{,}3 \cdot e^{-0{,}5x-0{,}6y} & \text{for } x, y \geq 0 \\ 0 & \text{ellers} \end{cases}$

Finner $F(a,b)$.

$F(a,b) = \begin{cases} \int_0^b \int_0^a f(x,y)\,dx\,dy & \text{for } a \geq 0 \text{ og } b \geq 0 \\ 0 & \text{ellers} \end{cases}$



Regner. NB: $0{,}3\,e^{-0{,}5x-0{,}6y} = \underbrace{0{,}3 \cdot e^{-0{,}5x}}_{a} \cdot e^{-0{,}6y}$

$$\int_0^a 0{,}3 \cdot e^{-0{,}5x-0{,}6y}\,dx = 0{,}3e^{-0{,}6y} \cdot \int_0^a e^{-0{,}5x}\,dx$$

Bruker substitusjon: $u = -0{,}5x$

$du = (-0{,}5x)'_x \cdot dx = -0{,}5 \cdot dx$

så $dx = \frac{1}{-0{,}5} \cdot du = -2\,du$

Altså $\int e^{-0{,}5x}\,dx = \int e^u \cdot (-2)\,du = -2\int e^u\,du = -2e^u + C$

$= -2e^{-0{,}5x} + C$

$$= 0{,}3e^{-0{,}6y} \cdot \left[-2e^{-0{,}5x} \right]_{x=0}^{x=a}$$

$$= 0{,}3e^{-0{,}6y} \cdot \left[-2e^{-0{,}5a} - (-2e^{-0{,}5\cdot 0}) \right]$$

$$= 0{,}3e^{-0{,}6y} \left[2 - 2e^{-0{,}5a} \right]$$

$$= 0{,}6\, e^{-0{,}6y} \left(1 - e^{-0{,}5a} \right)$$

Da er $F(a,b) = \int_0^b 0{,}6\, e^{-0{,}6y} \cdot (1 - e^{-0{,}5a})\, dy$

husk at y

$$= \cancel{0{,}6}(1-e^{-0{,}5a}) \cdot \left[\frac{1}{\cancel{-0{,}6}} \cdot e^{-0{,}6y} \right]_{y=0}^{y=b}$$

$$= (1-e^{-0{,}5a}) \left[-e^{-0{,}6b} - (-e^{-0{,}6\cdot 0}) \right]$$

($e^{-0{,}6\cdot 0} = 1$)

$$= \begin{cases} (1-e^{-0{,}5a})(1-e^{-0{,}6b}) & \text{for } a \geq 0 \text{ og } b \geq 0 \\ 0 & \text{ellers} \end{cases}$$

Skifter ofte variabler (etterpå): $a \rightsquigarrow x$, $b \rightsquigarrow y$

3. Marginalene

X og Y er simultant fordelte stok. var. med simultan tetthetsfunksjon $f(x,y)$. Men X og Y er stok. variabler hver for seg med tetthetsfunksjoner $f_X(x)$, $f_Y(y)$.

Vi har

$$f_X(x) = \int_{y=-\infty}^{y=\infty} f(x,y)\,dy \quad (\text{„integrerer bort } y\text{-ene"})$$

$$f_Y(y) = \int_{x=-\infty}^{x=\infty} f(x,y)\,dx \quad (\text{„integrerer bort } x\text{-ene"})$$

- kalles de marginale tetthetsfunksjonene til $f(x,y)$.

Eks
$$f(x,y) = \begin{cases} 0{,}3 \cdot e^{-0{,}5x} \cdot e^{-0{,}6y} & \text{for } x, y \geq 0 \\ 0 & \text{ellers} \end{cases}$$

Vil finne $f_X(x)$.

Ubestemt integral: $\int f(x,y)\,dy = 0{,}3e^{-0{,}5x} \int e^{-0{,}6y}\,dy$

$$= 0{,}3e^{-0{,}5x} \cdot \frac{1}{-0{,}6} e^{-0{,}6y} + C(x)$$

$$= -0{,}5\,e^{-0{,}5x} \cdot e^{-0{,}6y} + C(x)$$

Så $f_X(x) = \left[-0{,}5e^{-0{,}5x} \cdot e^{-0{,}6y}\right]_{y=0}^{y=\infty}$ (for $x \geq 0$)

$$= -0{,}5\, e^{-0{,}5x} \left[\lim_{y\to\infty} e^{-0{,}6y} - e^{-0{,}6\cdot 0} \right]$$

$\lim_{y\to\infty} e^{-0{,}6y}$ under the bracket; $e^{-0{,}6\cdot 0}$ under the bracket

$e^0 = 1$

$$\lim_{y\to\infty} \frac{1}{e^{0{,}6y}} = 0$$

Så

$$f_X(x) = \begin{cases} 0{,}5\, e^{-0{,}5x} & \text{for } x \geq 0 \\ 0 & \text{ellers} \end{cases}$$

Tilsvarende får vi $f_Y(y) = \begin{cases} \frac{0{,}3}{0{,}5} e^{-0{,}6y} \\ 0 \end{cases} = \begin{cases} 0{,}6\, e^{-0{,}6y}, & y \geq 0 \\ 0 & \text{ellers} \end{cases}$

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4. Forventning, varians og kovarians

Start: 16.03

X og Y er simultant fordelte stok. var. med tetthetsfunksjon $f(x,y)$.

Resultat (magisk i to variabler)

Anta $h(x,y)$ er en kontinuerlig funksjon i to variabler. Da er $Z = h(X,Y)$ en ny stok. var. med forventningsverdi

$$E(Z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) \cdot f(x,y)\, dx\, dy$$

Eks $E(X) \overset{\text{gammel def.}}{=} \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$

f_X ← marginalen $\int_{-\infty}^{\infty} f(x,y)\,dy$

Men ved det magiske resultatet i to variabler er det også slik:

$$E(X) \overset{\text{magisk}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f(x,y)\,dx\,dy$$

hvor $h(x,y) = x$ så

$$Z = h(X,Y) = X$$

$$E(Y) \overset{\text{magisk}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \cdot f(x,y)\,dx\,dy$$

hvor $h(x,y) = y$ så

$$Z = h(X,Y) = Y$$

Eks $E(X^2) \overset{\text{magisk}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 \cdot f(x,y)\,dx\,dy$

$h(x,y) = x^2$

(NB: Gammel definisjon $E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$)

Eks Fra $h(x,y) = x \cdot y$ får vi

$$E(X \cdot Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot f(x,y)\,dx\,dy$$

Definisjon Kovariansen til X og Y er

$$\mathrm{Cov}(X,Y) = E[(X-a)\cdot(Y-b)]$$ der

$a = E(X)$ og $b = E(Y)$.

Resultat $\mathrm{Cov}(X,Y) = E(X\cdot Y) - E(X)\cdot E(Y)$

Bevis: Se hefte.

Eks $f(x,y) = \begin{cases} x+y & \text{for } 0\le x\le 1 \text{ og } 0\le y\le 1 \\ 0 & \text{ellers} \end{cases}$

- tetheten til to simultant fordelte stok. var. X og Y. Vil finne $\mathrm{Cov}(X,Y)$.

Da trenger vi $E(X\cdot Y)$, $E(X)$ og $E(Y)$.

$$E(X) = \int_0^1\int_0^1 \underbrace{x^2+xy}_{x\cdot(x+y)}\, dx\, dy = \int_0^1 \left(\left[\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 y \right]_{x=0}^{x=1} \right) dy$$

$$= \int_0^1 \left(\tfrac{1}{3} + \tfrac{1}{2}\cdot y - 0 \right) dy = \int_0^1 \tfrac{1}{3} + \tfrac{1}{2}y\, dy$$

$$= \left[\tfrac{1}{3}y + \tfrac{1}{4}y^2 \right]_{y=0}^{y=1} = \tfrac{1}{3}\cdot 1 + \tfrac{1}{4}\cdot 1^2 - \left(\tfrac{1}{3}\cdot 0 + \tfrac{1}{4}\cdot 0^2\right) = \tfrac{1}{3}+\tfrac{1}{4}$$

$$= \frac{7}{12}$$

Tilsvarende er $E(Y) = \frac{7}{12}$ (ved symmetri)

$$E(X\cdot Y) = \int_0^1\int_0^1 \underbrace{x^2y + xy^2}_{xy\cdot(x+y)}\, dx\, dy = \int_0^1 \left(\left[\tfrac{1}{3}x^3 y + \tfrac{1}{2}x^2 y^2 \right]_{x=0}^{x=1} \right) dy$$

$$= \int_0^1 \left(\tfrac{1}{3}\cdot 1^3\cdot y + \tfrac{1}{2}\cdot 1^2\cdot y^2 - 0 \right) dy = \int_0^1 \tfrac{1}{3}y + \tfrac{1}{2}y^2\, dy$$

$$= \left[\frac{1}{3} \cdot \frac{1}{2} y^2 + \frac{1}{2} \cdot \frac{1}{3} y^3 \right]_{y=0}^{y=1} = \frac{1}{6} \left[y^2 + y^3 \right]_{y=0}^{y=1}$$

$$= \frac{1}{6} \left[1^2 + 1^3 - 0 \right] = \frac{2}{6} = \frac{1}{3}$$

Da er $\mathrm{Cov}(X, Y) = \frac{1}{3} - \frac{7}{12} \cdot \frac{7}{12} = \frac{48 - 49}{144}$ resultatet

$$= -\frac{1}{144}$$