	Emne	Lærebok
1	Introduksjon til kurset	
2	Vektorer og vektorregning	[E] 2.1 - 2.2, 2.6
3	Indreprodukt og lengden til vektorer	[E] 2.4

Oppgaver for Forelesning 1	
Oppgaver fra arbeidsboken	[DA] 1.1 - 1.11
Oppgaver fra læreboken	[E] 2.1 - 2.2, 2.4, 2.7 - 2.10, 2.12

(1) Matematikk valgfag:

Linear algebra F1-6
Sannsynlighetsregning F7-10
Differensial-ligninger F11-15

(2) Vektorer og vektorregning

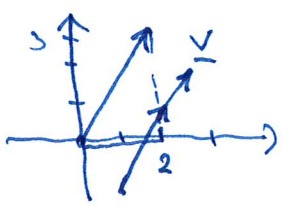
Def: En **n-vektor** er en kolonnevektor

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \vec{v}$$

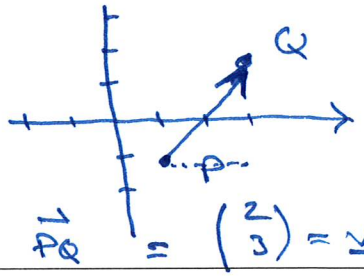
Eks: $\underline{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

2-vektor 3-vektor

Geometrisk representasjon: størrelse og retning



$P = (1, -1)$
$Q = (3, 2)$
$3-1, 2-(-1)$



$\vec{PQ} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \underline{v}$

Regning med vektorer:

* addisjon: $\underline{v} + \underline{w}$ ($\underline{v}, \underline{w}$ har samme antall komp., dvs begge er n-vektorer)

Eks: $\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 2+4 \\ 3+(-1) \end{pmatrix}$

$= \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

* subtraksjon: $\underline{v} - \underline{w}$ —— 11 ——

Eks: $\begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 2-4 \\ 3-(-1) \end{pmatrix}$

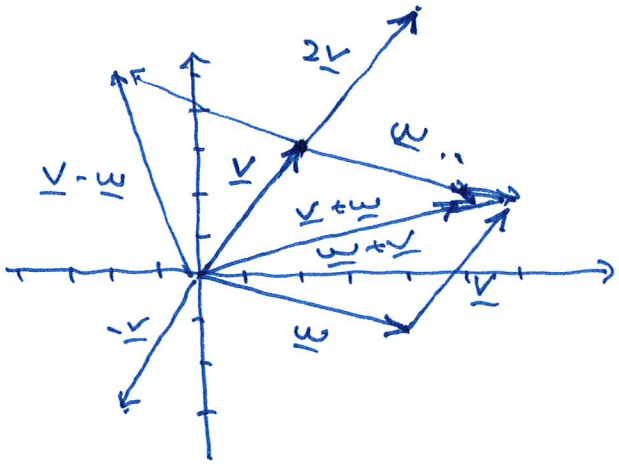
$= \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

* skalarmultiplikasjon: $r \cdot \underline{v}$

r: tall (skalar)

$\underline{v}$: vektor

Eks: $2 \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 2 \\ 2 \cdot 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ $\quad -1 \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

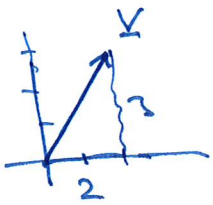


$\underline{v} - \underline{w} = \underline{v} + (-\underline{w})$

$\underline{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\quad \underline{w} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

Lengden til en vektor:

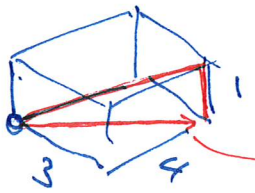
$n=2$: $\underline{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$



$$\|\underline{v}\| = \sqrt{2^2+3^2} = \sqrt{13}$$

$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$: $\|\underline{v}\| = \sqrt{v_1^2+v_2^2}$

$n=3$: $\underline{v} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$



$\sqrt{3^2+4^2} = 5$

$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$:

$\|\underline{v}\| = \sqrt{v_1^2+v_2^2+v_3^2}$

$\|\underline{v}\| = \sqrt{5^2+1^2} = \sqrt{26}$

$= \sqrt{\left(\sqrt{3^2+4^2}\right)^2+1^2}$

$= \sqrt{3^2+4^2+1^2}$

Generelt: $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \rightsquigarrow \|\underline{v}\| = \sqrt{v_1^2+v_2^2+v_3^2+\cdots+v_n^2}$

Merk: $\|\underline{v}\| \geq 0$ for alle vektorer $\underline{v}$ og

$\|\underline{v}\| = 0$ kun hvis $\underline{v} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \underline{0}$.

Lineærkombinasjoner av vektorer

Anta at $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$ er n-vektorer

Def:
En lineærkombinasjon av vektorene $\underline{v}_1, \ldots, \underline{v}_r$ er et uttrykk på formen

$$c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 + \ldots + c_r \cdot \underline{v}_r$$

der $c_1, c_2, \ldots, c_r$ er vilkårlige skalarer

Vi skriver:
$\text{span}(\underline{v}_1, \underline{v}_2, \ldots \underline{v}_r)$ for mengden av alle lin. komb. av $\underline{v}_1, \ldots \underline{v}_r$.

Eks:

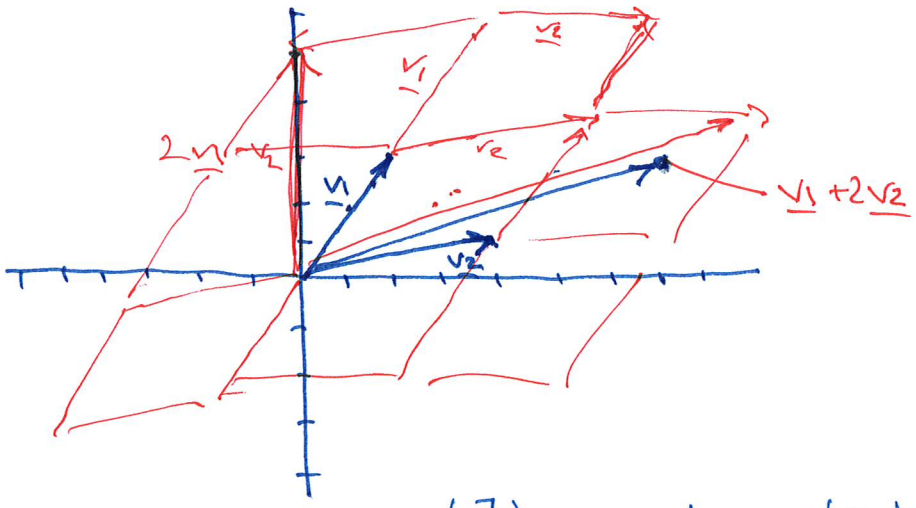
$\underline{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\underline{v}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

Linear komb. av $\underline{v}_1$ og $\underline{v}_2$:

Eks: $2\underline{v}_1 - 1 \cdot \underline{v}_2 = 2\underline{v}_1 - \underline{v}_2$ $\quad c_1 = 2 \quad c_2 = -1$

$\underline{v}_1 + 2\underline{v}_2$ * $\quad c_1 = 1 \quad c_2 = 2$



$$2\underline{v}_1 - \underline{v}_2 = \begin{pmatrix} 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

Er $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$ en linear komb. av $\underline{v}_1$ og $\underline{v}_2$?

$$c_1 \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

Vektorlikning.

$$\begin{pmatrix} 2c_1 \\ 3c_1 \end{pmatrix} + \begin{pmatrix} 4c_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 2c_1 + 4c_2 \\ 3c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

$$2c_1 + 4c_2 = 7 \quad | \cdot 3$$
$$3c_1 + c_2 = 3 \quad | \cdot 2$$

$$6c_1 + 12c_2 = 21$$
$$- \quad 6c_1 + 2c_2 = 6$$
$$10c_2 = 15$$
$$c_2 = \frac{15}{10} = 1.5$$

$$2c_1 + 4 \cdot 1.5 = 7$$
$$2c_1 + 6 = 7$$
$$2c_1 = 1$$
$$c_1 = 0.5$$

Konkl:

Svaret er ja! Dessuten: $\begin{pmatrix} 7 \\ 3 \end{pmatrix} = 0.5\,\underline{v_1} + 1.5\,\underline{v_2}$

Beskriv $\text{span}(\underline{v_1}, \underline{v_2})$:

La $\begin{pmatrix} a \\ b \end{pmatrix}$ være en vilkårlig 2-vektor

$$c_1 \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Er alle verdier av $\begin{pmatrix} a \\ b \end{pmatrix}$ har vektorlikningen en løsn (c_1, c_2).

$$2c_1 + 4c_2 = a \quad | \cdot 3$$
$$3c_1 + c_2 = b \quad | \cdot 2$$
$$10c_2 = 3a - 2b$$
$$c_2 = \frac{3a - 2b}{10}$$

Konkl: $\text{span}(\underline{v_1}, \underline{v_2}) = \mathbb{R}^2$

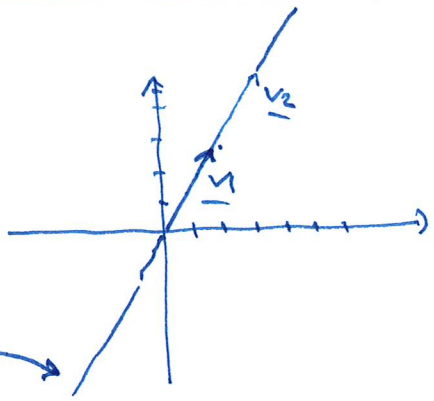
$\mathbb{R}^2$ = mengden av alle 2-vektorer

$$3c_1 + \frac{3a - 2b}{10} = b$$
$$3c_1 = \frac{10b}{10} - \frac{3a - 2b}{10}$$
$$3c_1 = \frac{-3a + 12b}{10}$$
$$c_1 = \frac{-3a + 12b}{30} = \frac{-a + 4b}{10}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0.3\,\underline{v_1} + 0.1\,\underline{v_2}$$

Eks: $\underline{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\underline{v}_2 = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$



$2\underline{v}_1 = \underline{v}_2$

$\text{span}(\underline{v}_1, \underline{v}_2) = \text{linjen} = \text{span}(\underline{v}_1)$

$c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 = c_1 \underline{v}_1 + c_2 \cdot 2\underline{v}_1 = (c_1 + 2c_2)\underline{v}_1$

Lineær uavhengighet:

La $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$ være n-vektorer

Eks: $\boxed{\underline{v}_2 = 2\underline{v}_1}$ betyr at $\underline{v}_1, \underline{v}_2$ er lineært avhengige

$(\underline{v}_1 = 0.5\,\underline{v}_2)$

Defn.
Vektorene $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$ kalles lineært avhengige hvis minst én av vektorene er en lin. komb. av de andre. Ellers kalles de lineært uavhengige.

Eks: i) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ og $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ er lineært uavhengige

ii) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ og $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ er lineært avhengige

Resultat: To vektorer er lineært avhengige hvis og bare hvis de ligger langs samme rette linje.

I tilfelle i): $x_1\underline{v}_1 + x_2\underline{v}_2 = \underline{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{matrix} a=0 \\ b=0 \end{matrix}$ $\Rightarrow$ $\begin{matrix} x_1 = 0 \\ x_2 = 0 \end{matrix}$

$\underline{v}_1, \underline{v}_2$ lin. uavh. $\Leftarrow$ kun den trivielle løsningen

Anta at én av vektorene er en lin. komb. av den andre, for eksempel $\underline{v}_1 = r\underline{v}_2 \Leftrightarrow \underline{v}_1 - r\underline{v}_2 = \underline{0} \Rightarrow (1, -r)$ er ikke-trivielt løsn.

③ Indreprodukt av vektorer (prikkprodukt, skalarprodukt)

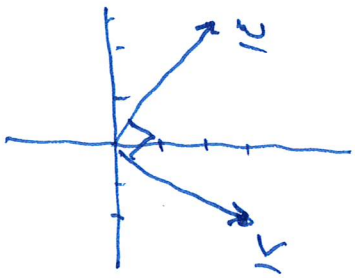
Eks: $\underline{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ $\Rightarrow$ $\underline{v}_1 \cdot \underline{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} = 2 \cdot 1 + 3 \cdot 4 = \underline{14}$

Generelt:

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} : \quad \underline{v} \cdot \underline{w} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = v_1 \cdot w_1 + v_2 \cdot w_2 + \dots + v_n \cdot w_n$$

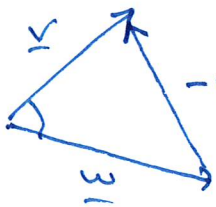
Defn: Vi sier at $\underline{v}$ og $\underline{w}$ er ortogonale ($\underline{v} \perp \underline{w}$) hvis $\underline{v} \cdot \underline{w} = 0$ $\Leftrightarrow$ vinkelen mellom $\underline{v}$ og $\underline{w}$ er 90°.

Eks: $\underline{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$: $\underline{v} \cdot \underline{w} = 3 \cdot 2 + (-2) \cdot 3 = 0$



Hvorfor?

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$



$$-\underline{w} + \underline{v} = \underline{v} - \underline{w} = \begin{pmatrix} v_1 - w_1 \\ v_2 - w_2 \end{pmatrix}$$

Vinkelen er 90° $\Leftrightarrow$ $\|\underline{v} - \underline{w}\|^2 = \|\underline{v}\|^2 + \|\underline{w}\|^2$

$$(v_1 - w_1)^2 + (v_2 - w_2)^2 = v_1^2 + v_2^2 + w_1^2 + w_2^2$$

$$v_1^2 - 2v_1w_1 + w_1^2 + v_2^2 - 2v_2w_2 + w_2^2 = \dots$$

$$-2v_1w_1 - 2v_2w_2 = 0 \quad | : (-2)$$

$$v_1w_1 + v_2w_2 = 0$$

$\Updownarrow$

$$\underline{v} \cdot \underline{w} = 0$$

Noen nyttige egenskaper (for indre produktet)

i) $\underline{v}\cdot\underline{v} = \|\underline{v}\|^2$ $\qquad \underline{v}=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$: $\underline{v}\cdot\underline{v} = v_1^2 + v_2^2$

ii) $\underline{v}\cdot\underline{w} = \underline{w}\cdot\underline{v}$

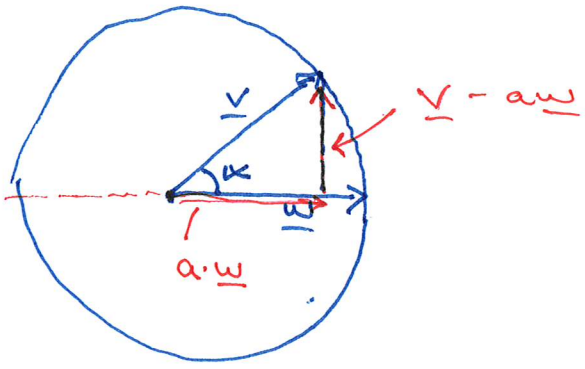
iii) $\underline{v}\cdot(r_1\underline{w}_1 + r_2\underline{w}_2) = r_1(\underline{v}\cdot\underline{w}_1) + r_2(\underline{v}\cdot\underline{w}_2)$

Eks: $(\underline{v}_1+\underline{v}_2)\cdot(\underline{w}_1-\underline{w}_2) = \underline{v}_1\cdot\underline{w}_1 + \underline{v}_1\cdot(-\underline{w}_2) + \underline{v}_2\cdot\underline{w}_1 + \underline{v}_2\cdot(-\underline{w}_2)$

$= \underline{v}_1\cdot\underline{w}_1 - \underline{v}_1\cdot\underline{w}_2 + \underline{v}_2\cdot\underline{w}_1 - \underline{v}_2\cdot\underline{w}_2$

Tolkning av prikkeproduktet

i) Tilfellet der $\|\underline{v}\| = \|\underline{w}\| = 1$: $\underline{v}\cdot\underline{w}$



$-1 \le a \le 1$

Vet at:

$a\underline{w}\cdot(\underline{v}-a\underline{w}) = 0$

$a(\underline{w}\cdot\underline{v}) - a\,\underline{w}\cdot\underline{w} = 0$

$(\underline{v}-a\underline{w})\cdot\underline{w} = 0$

$\underline{v}\cdot\underline{w} - a\cdot(\underline{w}\cdot\underline{w}) = 0$ ($\underline{w}\cdot\underline{w} = 1$)

$a = \underline{v}\cdot\underline{w}$

$\underline{v}\cdot\underline{w} = 1$: $\alpha = 0^\circ$

$\underline{v}\cdot\underline{w} = 0$: $\alpha = 90^\circ$

$\underline{v}\cdot\underline{w} = -1$: $\alpha = 180^\circ$

Sammenheng mellom α og a:

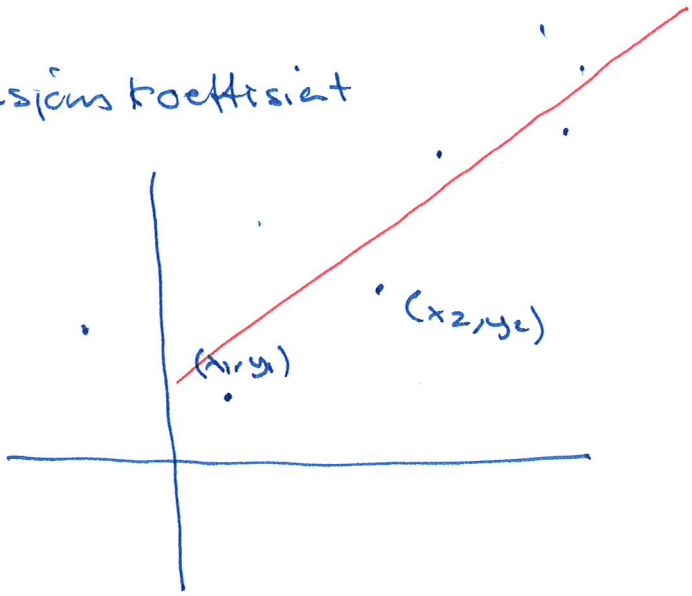
$a = \cos(\alpha)$

Generelt: $\|\underline{v}\|, \|\underline{w}\| > 0$

$-1 \le \dfrac{\underline{v}\cdot\underline{w}}{\|\underline{v}\|\cdot\|\underline{w}\|} \le 1$

Anvendelse: Korrelasjonskoeffisient

x	y
x_1	y_1
x_2	y_2
⋮	
x_N	y_N



data m to variabler og N observasjoner

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_N}{N}$$

$$\bar{y} = \frac{y_1 + y_2 + \dots + y_N}{N}$$

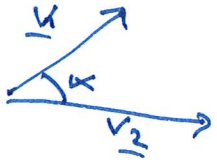
$$\underline{x} = \begin{pmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_N - \bar{x} \end{pmatrix} \qquad \underline{y} = \begin{pmatrix} y_1 - \bar{y} \\ y_2 - \bar{y} \\ \vdots \\ y_N - \bar{y} \end{pmatrix}$$

$$\underline{v}_1 = \frac{1}{\|\underline{x}\|} \cdot \underline{x} \qquad \underline{v}_2 = \frac{1}{\|\underline{y}\|} \underline{y}$$

$$r = \underline{v}_1 \cdot \underline{v}_2 = \frac{\underline{x} \cdot \underline{y}}{\|\underline{x}\| \cdot \|\underline{y}\|}$$ ← korrelasjons-koeff.

$$-1 \le r \le 1$$

$$= \cos(\alpha)$$



Eks: $\underline{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$$\|\underline{v}\| = \sqrt{3^2 + 4^2} = 5$$

$$\frac{1}{\|\underline{v}\|} \cdot \underline{v} = \frac{1}{5} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

$$(3/5)^2 + (4/5)^2 = \frac{3^2 + 4^2}{5^2} = 1$$