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① Repetisjon

$$\mathrm{Cov}(X,Y) = E\left[(X-\mu_X)\cdot(Y-\mu_Y)\right] = E(XY) - E(X)\cdot E(Y)$$

(μ_X, μ_Y)

Uavhengighet: X og Y uavhengige hvis

$$f(x,y) = f_X(x)\cdot f_Y(y)$$

X og Y er uavhengige $\Rightarrow \mathrm{Cov}(X,Y) = 0$

$$E(X+Y) = E(X) + E(Y)$$

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$$

$$E(aX+bY) = aE(X) + bE(Y)$$

$$\mathrm{Var}(aX+bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X,Y)$$

$$\mathrm{Var}(X) = E(X^2) - E(X)^2$$

(2) Betinget sannsynlighetsfordeling

Eks: (diskret fordeling)

Vi kaster to terninger, X = diff Y = max

X \ Y	1	2	3	4	5	6	
0	1/36	1/36	1/36	1/36	1/36	1/36	6/36
1	0	2/36	2/36	2/36	2/36	2/36	10/36
2	0	0	2/36	2/36	2/36	2/36	8/36
3	0	0	0	2/36	2/36	2/36	6/36
4	0	0	0	0	2/36	2/36	4/36
5	0	0	0	0	0	2/36	2/36
	1/36	3/36	5/36	7/36	9/36	11/36	1

$$P(Y=3 \mid X=2) = \frac{P(Y=3, X=2)}{P(X=2)} = \frac{2/36}{8/36} = \frac{1}{4}$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Anta X, Y kont stok. fordelte variabler

$$P(3 \le X \le 4 \mid Y = 40) = \lim_{h \to 0} \frac{P(3 \le X \le 4, 40 \le Y \le 40+h)}{P(40 \le Y \le 40+h)}$$

$$= \lim_{h \to 0} \frac{\int_3^4 \int_{40}^{40+h} f(x,y)\, dy\, dx}{\int_{40}^{40+h} f_Y(y)\, dy}$$

Ny stokastisk variabel:

$X \mid Y = 40$

Tetthetsfn:

$$f_{X \mid Y=40}(x) = \frac{f(x,40)}{f_Y(40)}$$

Minner om l'Hôpitals regel

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

$f(a)=0$, $g(a)=0$ "$\frac{0}{0}$"-uttrykk

Eks:

$$\lim_{x\to\infty}\frac{x}{e^x}\left(=\lim_{x\to\infty}\frac{e^{-x}}{-1/x}\right)$$

"$\frac{\infty}{\infty}$"-uttrykk

$$=\lim_{x\to\infty}\frac{1}{e^x}=0$$

Fundamentalteoremet for integrasjon

$$\int_0^h f(x)\,dx = \Big[F(x)\Big]_0^h = F(h)-F(0)$$ ← funksjon av h

$$(F(h)-F(0))'_h = F'(h) = f(h)$$

$$\boxed{\frac{d}{dh}\int_0^h f(x)\,dx = f(h)}$$

i) $$\frac{d}{dh}\int_{f_y(y)}^{y_0+h} \cancel{f(x,y)}\,dy = \cancel{f(x,y_0+h)}\cdot(y_0+h)'_h = \cancel{f(x,y_0+h)}$$ $f_y(y_0+h)$ $f_y(y_0+h)$

ii) $$\frac{d}{dh}\int_3^4\int_{y_0}^{y_0+h} f(x,y)\,dy\,dx = \int_3^4\left(\frac{d}{dh}\int_{y_0}^{y_0+h} f(x,y)\,dy\right)dx$$

$$= \int_3^4 f(x,y_0+h)\,dx$$

$$\lim_{h\to 0}\frac{\int_3^4 f(x,y_0+h)\,dx}{\cancel{f(x,y_0+h)}\;f_y(y_0+h)} = \frac{\int_3^4 f(x,y_0)\,dx}{\cancel{f(x,y_0)}\;f_y(y_0)} = \int_3^4 \frac{1}{f_y(y_0)}\cdot f(x,y_0)\,dx$$

$$= \int_3^4 \frac{f(x,y_0)}{f_y(y_0)}\,dx$$

Eks: X og Y kont stok var

Ny stok var: $X|Y=40$ tetthetsfn. $\frac{f(x,40)}{f_Y(40)}$

$$P(3 \le X \le 4 \mid Y=40) = \int_3^4 \frac{f(x,40)}{f_Y(40)}\,dx$$

Defn: $f_{X|Y=40}(x) = \frac{f(x,40)}{f_Y(40)}$ er tetthetsfn til $X|Y=40$

$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$ — " — $X|Y=y$

Eks: $E(X|Y=40) = \int_{-\infty}^{\infty} x \cdot f_{X|Y=40}(x)\,dx = \int_{-\infty}^{\infty} x \cdot \frac{f(x,40)}{f_Y(40)}\,dx$

Oppsummering: X og Y cts. fordelte stok. var. med tetthetsfu. $f(x,y)$

→ Ny stokastisk variabel $X|Y=y$ med tetthetsfu. $f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$

(gitt at $f_Y(y) \neq 0$)

Eks: $f(x,y) = x+y$, $0 \le x,y \le 1$

Marginal: $f_Y(y) = y + 1/2$ $(0 \le y \le 1)$

$X|Y=1/4$: $f_{X|Y=1/4}(x) = \frac{f(x,1/4)}{f_Y(1/4)}$

$= \frac{x+1/4}{1/2+1/4} \cdot \frac{4}{4} = \frac{4x+1}{3} = \frac{1}{3}(4x+1)$, $0 \le x \le 1$

$E(X|Y=1/4) = \int_0^1 x \cdot \frac{1}{3}(4x+1)\,dx = \frac{1}{3}\int_0^1 4x^2 + x\,dx$

$= \frac{1}{3}\left[\frac{4}{3}x^3 + \frac{1}{2}x^2\right]_0^1 = \frac{1}{3}\left(\frac{4}{3}+\frac{1}{2}\right) = \frac{4\cdot 2}{9\cdot 2} + \frac{1\cdot 3}{6\cdot 3}$

$= \frac{11}{18}$

$X|Y=1/2$: $f_{X|Y=1/2}(x) = \frac{f(x,1/2)}{f_Y(1/2)} = \frac{x+1/2}{1} = x+1/2$, $0 \le x \le 1$

$f_X(x) = x + 1/2$, $0 \le x \le 1$

$f_X(x) = \frac{f(x,1/2)}{f_Y(1/2)} \iff f_X(x) \cdot f_Y(1/2) = f(x,1/2)$

$E(X|Y=1/2) = \int_0^1 x \cdot (x+1/2)\,dx = \int_0^1 x \cdot f_X(x) = E(X)$

Merk: X og Y uavhengige $\iff f(x,y) = f_X(x) \cdot f_Y(y)$ for alle x,y

$\Uparrow$

$$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = f_X(x) \quad \text{for alle } x,y$$

(3) Kovariansmatrisen

$$\text{Var}(aX+bY) = a^2\text{Var}(X) + 2ab\,\text{Cov}(X,Y) + b^2\text{Var}(Y)$$

$$\underline{w} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad = (a\ b) \cdot \underbrace{\begin{pmatrix} \text{Var}(X) & \text{Cov}(X,Y) \\ \text{Cov}(X,Y) & \text{Var}(Y) \end{pmatrix}}_{\Sigma} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \underline{w}^T \Sigma \underline{w}$$

Kovariansmatrisen:

$$\Sigma = \begin{pmatrix} \text{Var}(X) & \text{Cov}(X,Y) \\ \text{Cov}(X,Y) & \text{Var}(Y) \end{pmatrix}$$

Kovariansmatrisen er positiv semidefinitt. Siden $\text{Var}(aX+bY) \geq 0$

Utvidelse til n simultant fordelte stokastiske variable: $X_1, X_2, \ldots, X_n$

$X = w_1X_1 + w_2X_2 + \ldots + w_nX_n$:

$$\text{Var}(X) = w_1^2\text{Var}(X_1) + 2w_1w_2\,\text{Cov}(X_1,X_2) + \ldots + 2w_1w_3\,\text{Cov}(X_1,X_3) + \ldots$$

$$= \underline{w}^T \Sigma \underline{w} \qquad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \text{Var}(X_1) & \text{Cov}(X_1,X_2) & \cdots \\ \text{Cov}(X_1,X_2) & \text{Var}(X_2) & \cdots \\ \vdots & \vdots & \ddots \\ & & \text{Var}(X_n) \end{pmatrix}$$

$$E(X) = w_1E(X_1) + \ldots + w_nE(X_n)$$

$$= (E(X_1)\ E(X_2) \ldots E(X_n)) \cdot \underline{w}$$

i) min $\underbrace{Var(X)}_{f(\underline{w})} = \underline{w}^T \Sigma \underline{w}$, Σ positiv semidefin

$f'(\underline{w}) = 2\Sigma\underline{w} = \underline{0}$

$\underline{w} = \underline{0}$

Globalt min

$Var(X) = 0$ hvis $\underline{w} = \underline{0}$

ii) min $Var(X) = \underline{w}^T \Sigma \underline{w}$ når $\underbrace{w_1 + w_2 + \dots + w_n = 1}$

$(1\ 1\ \dots\ 1)\cdot\underline{w} = 1$

$B\cdot\underline{w} = 1$

$B = (1\ 1\ \dots\ 1)$

$n=3$:

min $f(w_1,w_2,w_3) = \underline{w}^T\Sigma\underline{w}$ når $\underbrace{w_1+w_2+w_3=1}$

$B\cdot\underline{w} = 1$ $\quad B = (1\ 1\ 1)$

$L = \underline{w}^T\Sigma\underline{w} - \lambda(B\underline{w} - 1)$

Σ pos. semidefin

$\lambda_1, \lambda_2, \lambda_3 \geq 0$

$|\Sigma| = \lambda_1\cdot\lambda_2\cdot\lambda_3$

$L'_{w_1} = 0$
$L'_{w_2} = 0$
$L'_{w_3} = 0$

$L'(\underline{w}) = 2\Sigma\underline{w} - \lambda(B^T) = \underline{0}$

$2\Sigma\underline{w} = \lambda\cdot B^T$

$\Sigma\underline{w} = \frac{\lambda}{2}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}$

$B\cdot\underline{w} = 1$

$|\Sigma| \neq 0$: (Σ pos. definit)

$\underline{w} = \Sigma^{-1}\frac{\lambda}{2}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{\lambda}{2}\cdot\Sigma^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix}$

$(1\ 1\ 1)\cdot\frac{\lambda}{2}\Sigma^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix} = 1$

$\frac{\lambda}{2}(1\ 1\ 1)\Sigma^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix} = 1$

$\lambda = \frac{2}{(1\ 1\ 1)\Sigma^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix}}$

$$\Rightarrow \frac{x}{2} = \frac{1}{(1\,1\,1)\Sigma^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix}} \Rightarrow w = \frac{1}{(1\,1\,1)\Sigma^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix}} \cdot \Sigma^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Ex: $\Sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 11 \end{pmatrix}$ $\quad |\Sigma| = 1\cdot(6) - 2(1) + 3(-1) = 1$

$$\Sigma^{-1} = \frac{1}{1}\begin{pmatrix} 6 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}^T = \begin{pmatrix} 6 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\Sigma^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix} 6 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}4\\0\\-1\end{pmatrix}$$

$$(1\;1\;1)\,\Sigma^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix} = (1\;1\;1)\cdot\begin{pmatrix}4\\0\\-1\end{pmatrix} = 3$$

min portefølje: $w = \frac{1}{3}\cdot\begin{pmatrix}4\\0\\-1\end{pmatrix} = \begin{pmatrix}4/3\\0\\-1/3\end{pmatrix}$

$$\text{Var} = (4/3\;\;0\;\;-1/3)\cdot\begin{pmatrix}1&2&3\\2&5&7\\3&7&11\end{pmatrix}\cdot\begin{pmatrix}4/3\\0\\-1/3\end{pmatrix} = 1/3$$

Merk: i) Hvis $|\Sigma| = 0$ (Σ ikke positiv defin.), så betyr det at en av x_i'ene er en lin. komb. av de andre.

ii) Hvis $|\Sigma| \neq 0$ (Σ pos. defin), så er også Σ^{-1} symmetrisk og pos. defin.

$$\Sigma x = \lambda x \quad (\lambda > 0)$$

$$x = \Sigma^{-1}\Sigma x = \Sigma^{-1}\lambda x = \lambda \Sigma^{-1} x \quad | : \lambda$$

$$\frac{1}{\lambda} x = \Sigma^{-1} \cdot x \quad \text{dvs } 1/\lambda \text{ egenverdi for } \Sigma^{-1}$$

$1/\lambda > 0 \Rightarrow \Sigma^{-1}$ pos. defin.

$\Rightarrow (1\,1\,1)\Sigma^{-1}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} > 0$