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① Repetisjon

X og Y simultant fordelte tetthets fn. $f(x,y)$ $\rightsquigarrow$ $X|Y=y$ stokastisk variabel med tetthetsfunksjon

$$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$$

når $f_Y(y) \neq 0$.

Eks: $E[X|Y=y] = \int_{-\infty}^{\infty} x \cdot \frac{f(x,y)}{f_Y(y)}\, dx$

Kovariansmatrisen:

$X_1, X_2, \ldots, X_n$ simultant fordelte

$Z = w_1X_1 + w_2X_2 + \ldots + w_nX_n$ "portefølje" ($w_1, w_2, \ldots, w_n$ konst.)

$E(Z) = w_1E(X_1) + w_2E(X_2) + \ldots + w_nE(X_n)$

$Var(Z) = w_1^2 Var(X_1) + 2w_1w_2\,Cov(X_1,X_2) + \ldots + w_n^2 Var(X_n)$

$= \underline{w}^T \Sigma \underline{w}$, $\underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$ $\Sigma = \begin{pmatrix} Var(X_1) & Cov(X_1,X_2) & \cdots \\ Cov(X_1,X_2) & Var(X_2) & \cdots \\ \vdots & \vdots & \ddots \\ & & Var(X_n) \end{pmatrix}$

kovariansmatrise (positiv semidef.)

[DA] 9.3 $f(x,y) = \begin{cases} e^{k(x+y)} & , x,y \geq 0 \\ 0 & , \text{ellers} \end{cases}$

a) Tetthetsfkn: i) $f(x,y) \geq 0$ Vok

ii) $\int_0^\infty \int_0^\infty e^{k(x+y)} \, dy \, dx = 1$

$e^{kx} \cdot e^{ky}$

Spesielt: $\int_0^\infty e^{kx+ky} \, dy = \int e^u \frac{1}{k} \, du = \frac{1}{k} \left[e^{kx+ky} \right]_0^\infty$

$u = kx+ky$, $du = k \cdot dy$

$= \frac{1}{k} e^{kx} \lim_{y \to \infty} e^{ky} - \frac{1}{k} e^{kx} \cdot 1 = \begin{cases} \infty, & k > 0 \\ -\frac{1}{k} e^{kx}, & k < 0 \end{cases}$

$k<0$: $\int_0^\infty -\frac{1}{k} e^{kx} \, dx = -\frac{1}{k} \left[\frac{1}{k} e^{kx} \right]_0^\infty = \left(-\frac{1}{k^2} \lim_{x \to \infty} e^{kx} \right)$

$- \left(-\frac{1}{k^2} \cdot 1 \right) = \frac{1}{k^2} - 0 = \frac{1}{k^2} = 1$

$k^2 = 1$

$k = -1$

Konkl: $f(x,y) = e^{-(x+y)}$, $x,y \geq 0$

$f_X(x) = \int_{-\infty}^{\infty} f(x,y) \, dy = \int_0^\infty e^{-x-y} \, dy = \underline{\underline{e^{-x}}}$, $x \geq 0$

b) $E(X) = \int_0^\infty x e^{-x} \, dx = \left[-x e^{-x} \right]_0^\infty - \int_0^\infty -e^{-x} \cdot 1 \, dx = \left[-x e^{-x} - e^{-x} \right]_0^\infty$

$u = -e^{-x}$ $v = x$

$u' = e^{-x}$ $v' = 1$

$= \lim_{x \to \infty} \frac{-x-1}{e^x} + 1 = \underline{\underline{1}}$

$$Var(x) = E(x^2) - E(x)^2 = E(x^2) - 1^2 = 2 - 1 = \underline{1}$$

$$E(x^2) = \int_0^\infty x^2 \cdot e^{-x}\,dx = -x^2 e^{-x} - \int -2xe^{-x}\,dx$$

$$\left| \begin{array}{ll} u = -e^{-x} & v = x^2 \\ u' = e^{-x} & v' = 2x \end{array} \right|$$

$$= \left[-x^2 e^{-x} + 2(-xe^{-x} - e^{-x}) \right]_0^\infty = \lim_{x\to\infty} \frac{-x^2 - 2x - 2}{e^x} - (-2)$$

$$= 0 + 2 = \underline{2}$$

c) Uavhengighet: $f_X(x) \cdot f_Y(y) = f(x,y)$ $\quad x, y \geq 0$

$e^{-x} \quad e^{-y} \quad e^{-x-y}$

(Bh)

X, Y uavhengige

d) $E(XY) = E(X) \cdot E(Y) = 1 \cdot 1 = \underline{1}$

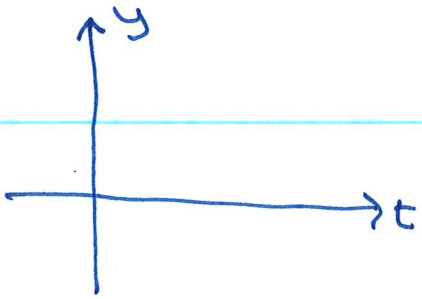
$Cov(X,Y) = 0$ siden X, Y uavh.

$E(XY) - E(X) \cdot E(Y)$

$$E(XY) = \int_0^\infty \int_0^\infty xy \cdot e^{-x-y}\,dy\,dx$$

(1) Første ordens differensialligninger

Defn. En differensialligning er en ligning der den ukjente er en funksjon $y = y(t)$ og der de deriverte til y inngår. Den kalles **første ordens** hvis y' inngår men ikke høyere ordens deriverte.



Ex: $y' = 12t - 6$ $\quad (y'(t) = 12t - 6)$

$y' = 2y + 12$ $\quad (y'(t) = 2y(t) + 12)$

Std. form: $y' = F(y,t)$ modell for veksten til y

Eks: $y' = 12t - 6 \quad | \int \ldots dt$

$$\int y' dt = \int 12t - 6\, dt$$

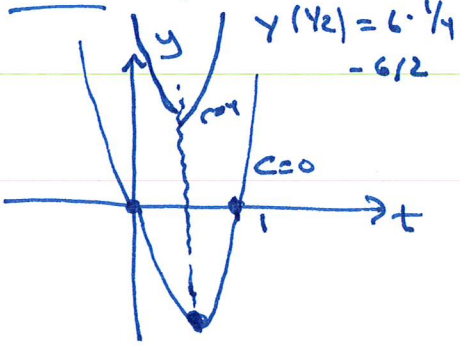
$$y = 12 \cdot \tfrac{1}{2} t^2 - 6t + C$$

$$y = 6t^2 - 6t + C$$

generell løsning (inneholder alle løsn. av $y' = 12t - 6$)

$t = 0$: $y(0) = C$ initialbetingelse

$C = 0$: $y = 6t^2 - 6t$



$C = 4$: $y = 6t^2 - 6t + 4$

$y' = 12t - 6$ ⟶ $y = 6t^2 - 6t + C$

$y' = 12t - 6$, $y(0) = 4$:

$y = 6t^2 - 6t + C$

$y(0) = 4$: $4 = 6 \cdot 0^2 - 6 \cdot 0 + C$

$C = 4$

$\Rightarrow y = 6t^2 - 6t + 4$ partikulær løsn.

Resultat: En førsteordens diff.likn. har generell løsn. som avhenger av en ubestemt konstant/parameter

Enkleste tilfelle: $y' = F(t)$ løsbar ved enkel integration

$y = \int F(t)\,dt$

(3) Separable differensialligninger

Defn: En første ordens diff.likn. kalles separabel om den kan skrives

$$y' = f(t) \cdot g(y)$$

Eks: $y' = 12t - 6$ er separabel ✓

$(12t - 6) \cdot 1 = (2t - 1) \cdot 6$

$y' = 2y + 12$ er separabel ✓

$= 2 \cdot (y + 6)$

$f(t)$ $g(y)$

$\int y'\,dt = \int 2y + 12\,dt$

Eks: $y' = 2y - t$ ikke separabel

$= (y \pm \ldots)(t \pm \ldots)$

$y' = 5y - y^2$ separabel

$= 1 \cdot (5y - y^2)$

$f(t)$ $g(y)$

Metode for å løse separable diff.-likninger

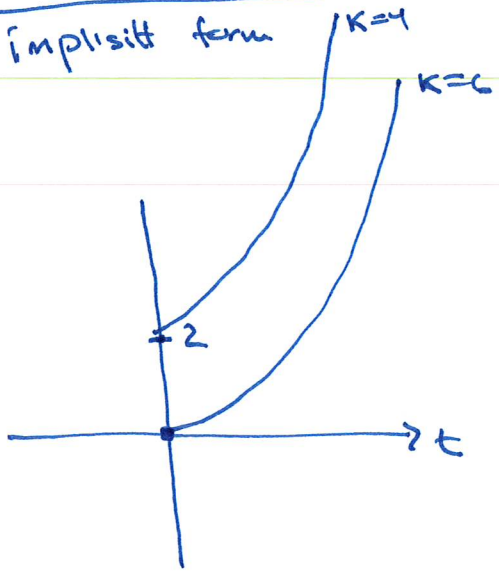
$y' = f(t) \cdot g(y) \quad | : g(y)$

$\frac{1}{g(y)} y' = f(t) \quad | \int \ldots dt$

$\int \frac{1}{g(y)} y' \, dt = \int f(t) \, dt$ ← variablene er separert

$$\boxed{\int \frac{1}{g(y)} \, dy = \int f(t) \, dt}$$

implisitt form



$y(0) = K - 6$

$y' = 2y + 12 = 2 \cdot (y + 6) \quad | : (y+6)$

$\frac{1}{y+6} y' = 2 \quad | \int \ldots dt$

$\int \frac{1}{y+6} y' \, dt = \int 2 \, dt$

$= 2t + C$

$\int \frac{1}{y+6} \, dy$

Substitusjon: $y = y(t)$

$dy = y'(t) \, dt$

$\boxed{dy = y' \, dt}$

$\ln|y+6| = 2t + C_2 \quad + C_1$

$\ln|y+6| = 2t \quad + C_2 - C_1$

$\ln|y+6| = 2t + C$ generell løsn. på implisitt form

$e^{\ln|y+6|} = e^{2t+C}$

$|y+6| = e^{2t} \cdot e^C$

$y + 6 = \pm e^{2t} \cdot e^C = Ke^{2t}$

$y = Ke^{2t} - 6$ generell løsn. eksplisitt form

Eks: $y' = ty \quad | : y$

$\frac{1}{y} y' = t$

$\int \frac{1}{y} y' dt = \int t\,dt$

$\int \frac{1}{y} dy = \int t\,dt$

$\ln|y| = \frac{1}{2}t^2 + C \quad | \exp$

$e^{\ln|y|} = e^{\frac{1}{2}t^2 + C} = e^{\frac{1}{2}t^2} \cdot e^C$

$|y| = e^C \cdot e^{\frac{1}{2}t^2}$

$y = \underbrace{\pm e^C}_{K} \cdot e^{\frac{1}{2}t^2} \qquad y = K \cdot e^{\frac{1}{2}t^2}$

Eks: $y' = 5y - 10y^2 = 5 \cdot (y - 2y^2) \quad | : y - 2y^2$

$\frac{1}{y - 2y^2} y' = 5$

$\int \frac{1}{y - 2y^2} dy = \int 5\,dt$

$\frac{1}{y \cdot (1-2y)} = \frac{A}{y} + \frac{B}{1-2y} \quad | \cdot y(1-2y)$

$1 = A(1-2y) + By$

$= A - 2Ay + By = (B - 2A) \cdot y + A$

$A = 1 \qquad B - 2A = 0$

$B = 2$

$\int \frac{1}{y} + \frac{2}{1-2y} dy = \int 5\,dt$

$\ln|y| - \ln|1-2y| = 5t + C$

$\ln \frac{|y|}{|1-2y|} = 5t + C \quad | \exp$

$\left|\frac{y}{1-2y}\right| = e^{5t+C} = e^C \cdot e^{5t}$

$\frac{y}{1-2y} = K e^{5t} \quad | \cdot (1-2y)$

$y = (1-2y) \cdot K e^{5t}$

$y + y \cdot 2K e^{5t} = K e^{5t}$

$$y = \frac{K \cdot e^{5t}}{1 + 2Ke^{5t}} \quad \leftarrow \quad \frac{y(1+2Ke^{5t})}{1+2Ke^{5t}} = \frac{Ke^{5t}}{1+2Ke^{5t}}$$

Eks (Eksamen V2017, 3c)

Løs $y' = 2te^{-y}$, $y(0) = 0$

$$y' = 2t \cdot e^{-y} \quad | : e^{-y} = \cdot e^{y}$$

$$e^{y} \cdot y' = 2t$$

$$\int e^{y} y' dt = \int 2t\, dt$$

$$\int e^{y}\, dy = t^2 + C$$

$$e^{y} = t^2 + C$$

$y(0) = 0$:

$t = 0$, $y = 0$: $e^{0} = 0^2 + C$

$C = 1$

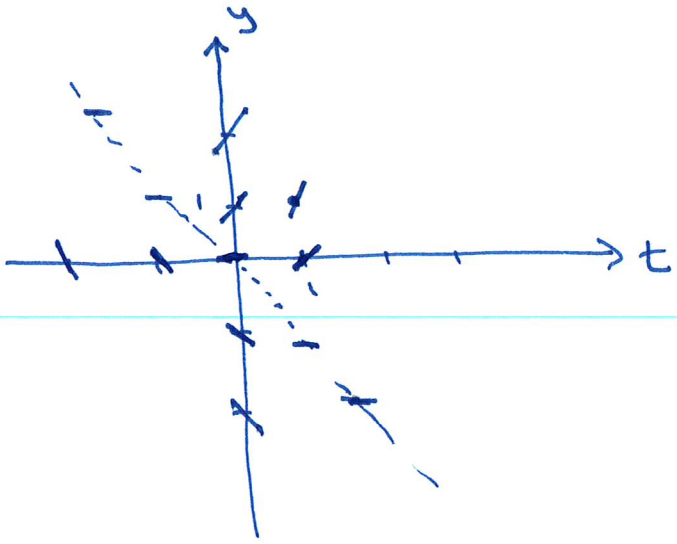
$$e^{y} = t^2 + 1 \quad | \ln(\cdot)$$

$$y = \ln(t^2 + 1)$$

Fasediagram:

Eks: $y' = t + y$

Tegn inn stigningstallet

$y' = F(t,y)$

for mange punkter (t,y).