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① Repetisjon:

Første ordens lineær diff.likn: $\boxed{y' + a(t)\,y = b(t)}$

Metode I: Integrerende faktor (generell metode)

Integrerende faktor:

$u = e^{\int a(t)dt}$: $\quad y' + a(t)\,y = b(t) \mid \cdot u(t)$

$(y \cdot u(t))' = b(t) \cdot u(t)$

$y \cdot u(t) = \int b(t)\,u(t)\,dt$

$y = \frac{1}{u(t)} \cdot \int b(t)\,u(t)\,dt$

Metode 2: Superposisjon $\quad \boxed{y' + a(t)\,y = b(t)}$

$D = \frac{d}{dt} + a(t)\cdot$ differensialoperator

$D(y) = y' + a(t)\cdot y$

lineær $\leftrightarrow$ $\begin{cases} D(y_1 + y_2) = D(y_1) + D(y_2) \\ D\cdot(c\cdot y_1) = c\cdot D(y_1) \end{cases}$

$\Downarrow$

superposisjon: $y = y_h + y_p$ $\begin{cases} y_h: \text{generell løsn. av den homogene} \\ y_p: \text{en spesiell løsn.} \end{cases}$

Eks: $y' - 3y = t$ $a(t) = -3$, $b(t) = t$ lin. første ordens

i) $\int -3\,dt = -3t + C$, $u = e^{-3t}$

$$(y \cdot e^{-3t})' = te^{-3t}$$

$$y \cdot e^{-3t} = \int te^{-3t}\,dt$$

$u = -\frac{1}{3}e^{-3t}$ $v = t$
$u' = e^{-3t}$ $v' = 1$

$$= -\frac{1}{3}te^{-3t} - \int -\frac{1}{3}e^{-3t}\,dt$$

$$= -\frac{1}{3}te^{-3t} + \frac{1}{3}\left(-\frac{1}{3}e^{-3t}\right) + C$$

$$y \cdot e^{-3t} = \left(-\frac{1}{3}t - \frac{1}{9}\right)e^{-3t} + C \quad | \cdot e^{3t}$$

$$y = Ce^{3t} - \frac{1}{3}t - \frac{1}{9}$$

(y_h og y_p)

ii) $Y = y_h + y_p$

$$= C \cdot e^{3t} - \frac{1}{3}t - \frac{1}{9}$$

y_h: $y' - 3y = 0$

Kar. likn. $r - 3 = 0$ $r = 3$

$$y_h = C \cdot e^{3t}$$

y_p: $y' - 3y = t$ ← $\begin{cases} y = At + B \\ y' = A \end{cases}$

$$y_p = -\frac{1}{3}t - \frac{1}{9}$$

$A - 3(At + B) = t$

$(-3A)t + (A - 3B) = 1 \cdot t + 0$

$-3A = 1$ $A = -1/3$

$A - 3B = 0$ $-1/3 - 3B = 0$ $B = -1/9$

Alt. for y_p:

$y = t$: $D(t) = 1 - 3t$

$y = 1$: $D(1) = 0 - 3 = -3$

$D\left(-\frac{1}{3}t\right) = -\frac{1}{3} + t$

$D\left(-\frac{1}{9} \cdot 1\right) = -\frac{1}{9}(-3) = \frac{1}{3}$

$D\underbrace{\left(-\frac{1}{3}t - \frac{1}{9}\right)}_{y_p} = t$

(2) Andre ordens lineære diff. likninger

Eks: Andre ordens diff.likning

$$y'' = 12 \quad | \int _\, dt$$

$$y' = \int 12\,dt = 12t + C \quad | \int _\, dt$$

$$y = \int 12t + C\,dt = \underline{6t^2 + Ct + D}$$

generell løsn med to ubestemte parametre

$y'' = 12$, $y(0) = 6$, $y'(0) = 3$

to start betingelser

$y = 6t^2 + Ct + D$

$y(0) = 6$: $6 = D$ $\quad \underline{D = 6}$

$y'(0) = 3$: $C = 3$ $\quad \underline{C = 3}$

$$\Downarrow$$

$$y = \underline{6t^2 + 3t + 6}$$

Defn: En andre ordens diff.likn. er lineær hvis den kan skrives

$$y'' + a(t) \cdot y' + b(t) \cdot y = h(t)$$

Den kalles homogen hvis $h(t) = 0$, inhomogen ellers.

Den har konstante koeff. hvis $a(t) = a$ og $b(t) = b$ er konstanter.

Merk: $D(y) = y'' + a(t)y' + b(t)y$ er en lineær diff. operator

siden i) $D(y_1 + y_2) = D(y_1) + D(y_2)$

ii) $D(c \cdot y_1) = c \cdot D(y_1)$

Sjekk: i) $D(y_1+y_2) = (y_1+y_2)'' + a(t)\cdot(y_1+y_2)' + b(t)(y_1+y_2)$

$= y_1'' + y_2'' + a(t)(y_1' + y_2') + b(t)(y_1+y_2)$

$= D(y_1) + D(y_2)$ vok.

ii) $D(cy_1) = (cy_1)'' + a(t)(cy_1)' + b(t)(cy_1)$

$= cy_1'' + a(t)\cdot cy_1' + b(t)\cdot c\cdot y_1$

$= c\,D(y_1)$ vok.

Ex: $y'' - 7y' + 12y = 4$ inhomogen konstante koeff.

$Y = y_h + y_p =$

y_p: $y'' - 7y' + 12y = 4$ $\quad h(t) = 4$, $h'(t) = 0$, $h''(t) = 0$ $\quad \begin{cases} Y = A \\ Y' = 0 \\ Y'' = 0 \end{cases}$

$0 - 7(0) + 12(A) = 4$

$12A = 4$

$A = 4/12 = 1/3$ $\quad y_p = 1/3$

Metode: Varierende parametre

Velg en funksjon $y(t)$ med parametre, av samme type som $h(t)$ ($h'(t)$, $h''(t)$).

Sett inn i diff.likningen, og juster parametrene for å finne en løsning.

y_h: $y'' - 7y' + 12y = 0$

Metode: Karakteristiske likning (når $a(t)=a$ og $b(t)=b$ er konstante koeff.)

$y'' \mapsto r^2 \quad y' \mapsto r \quad y \mapsto 1$

$r^2 - 7r + 12 = 0$

$(r-3)(r-4) = 0$

$r=3$ eller $r=4$

eller

$$r = \frac{7 \pm \sqrt{(-7)^2 - 4\cdot 12}}{2\cdot 1} = \frac{7 \pm 1}{2} = 3, 4$$

$$\Rightarrow y_h = C_1 e^{3t} + C_2 e^{4t}$$

Generelt: Homogent tilfelle, konstante koeff.

$y'' + ay' + by = 0$

Kar.likn: $r^2 + ar + b = 0$

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$\Delta = a^2 - 4b > 0$: $y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ $(r_1 \neq r_2)$

$\Delta = a^2 - 4b = 0$: $y_h = C_1 e^{r_1 t} + C_2 t e^{r_1 t}$ $(r_1 = r_2 = -a/2)$

$\Delta = a^2 - 4b < 0$: $y_h = e^{-a/2} \cdot (C_1 \cdot \cos \beta t + C_2 \cdot \sin \beta t)$

$r = -\frac{a}{2} \pm \underbrace{\frac{\sqrt{4b-a^2}}{2}}_{\beta} \cdot \sqrt{-1}$

Forklaring: Karakteristisk likning

$$y'' + ay' + by = 0 \quad \rightsquigarrow \quad r^2 + ar + b = 0$$

Ide: Tester når $y = e^{rt}$ er løsning.

$$\begin{cases} y = e^{rt} \\ y' = e^{rt} \cdot r \\ y'' = e^{rt} \cdot r^2 \end{cases}$$

$$\left(r^2 e^{rt}\right) + a\left(re^{rt}\right) + b\left(e^{rt}\right) = 0$$

$$e^{rt} \cdot (r^2 + ar + b) = 0$$

$$r^2 + ar + b = 0$$

Konkl: Løsningen av $r^2 + ar + b = 0$ er de verdiene av r slik at $y = e^{rt}$ er løsn. av diff.likn.

Anta at $\Delta = a^2 - 4b = 0 \iff r_1 = r_2 = -a/2$:

$r_1 = -a/2$ er karakteristisk rot $\Rightarrow C_1 \cdot e^{-a/2 \cdot t}$ er løsn. av diff.likn.

Prøver: $h(t) \cdot e^{rt} = y = (At + B)e^{rt}$

$$y' = h'(t) \cdot e^{rt} + h(t) \cdot e^{rt} \cdot r = \left(h'(t) + rh(t)\right) \cdot e^{rt}$$

$$y'' = \left(h''(t) + rh'(t)\right) e^{rt} + \left(h'(t) + rh(t)\right) \cdot e^{rt} \cdot r$$

$$= \left(h''(t) + rh'(t) + rh'(t) + r^2 h(t)\right) \cdot e^{rt}$$

$$\left(h'' + 2rh' + r^2 h\right) e^{rt} + a\left(h' + rh\right) e^{rt} + b\left(h e^{rt}\right) = 0$$

$$h'' + 2rh' + r^2 h + ah' + arh + bh = 0$$

$$h \cdot \underbrace{\left(r^2 + ar + b\right)}_{=0} + \underbrace{\left(h'' + 2rh' + ah'\right)}_{(2r+a)h' = 0} = 0$$

$h'' = 0$

$h = At + B$

Eksempler: $y'' - 4y' + 4y = 1 + 2e^t$

$y = y_h + y_p = (C_1 + C_2 t)e^{2t} + 1/4 + 2e^t$

y_h: $y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$ $\quad r_1 = r_2 = 2$ $\quad y_h = C_1 e^{2t} + C_2 t e^{2t}$

$r = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 4}}{2} = 2 \pm 0$ $\quad = (C_1 + C_2 t)e^{2t}$

y_p: $y'' - 4y' + 4y = 1 + 2e^t$

$h(t) = 1 + 2e^t$

$h'(t) = 2e^t$

$\S$

$y = A + Be^t$

$y' = Be^t$

$y'' = Be^t$

$(Be^t) - 4(Be^t) + 4(A + Be^t) = 1 + 2e^t \leftarrow = 1 + 2e^t$

$4A + B \cdot e^t$

$4A = 1 \quad A = 1/4$

$B = 2 \quad B = 2$

$y_p = 1/4 + 2e^t$

Eks: $y'' - 4y' + 4y = 3e^{2t}$ $\quad y = y_h + y_p = (C_1 + C_2 t)e^{2t} +$

y_p: $h(t) = 3e^{2t}$

$h' = 3 \cdot e^{2t} \cdot 2 = 6e^{2t}$

$h'' = 6e^{2t} \cdot 2 = 12e^{2t}$

$y = A \cdot e^{2t}$

$y' = 2Ae^{2t}$

$y'' = 4Ae^{2t}$

$(4Ae^{2t}) - 4(2Ae^{2t}) + 4(Ae^{2t}) = 3e^{2t}$

$0A \cdot e^{2t} = 3e^{2t}$

Prøver å multiplisere med t:

$y = At\,e^{2t}$

$y' = A\cdot e^{2t} + At\cdot 2e^{2t} = (A+2At)\,e^{2t}$

$y'' = 2Ae^{2t} + (A+2At)\cdot 2e^{2t} = (4A+4At)\,e^{2t}$

$\int$

$(4A+4At)e^{2t} - 4(A+2At)e^{2t} + 4(Ate^{2t}) = 3e^{2t}$

$(\cancel{4A} - \cancel{4A} + \cancel{4At} - \cancel{8At} + \cancel{4At})\,e^{2t} = 3e^{2t}$

$y = At^2\cdot e^{2t}$

$y' = 2At\,e^{2t} + At^2\cdot 2e^{2t} = (2At + 2At^2)e^{2t}$

$y'' = (2A+4At)e^{2t} + (2At+2At^2)\cdot 2e^{2t} = (2A+8At+4At^2)e^{2t}$

$\int$

$(2A+\cancel{8At}+\cancel{4At^2})e^{2t} - 4(\cancel{2At}+\cancel{2At^2})e^{2t} + \cancel{4At^2}e^{2t} = 3e^{2t}$

$(2A)e^{2t} = 3e^{2t} \qquad 2A = 3 \qquad A = 3/2 \qquad y_p = \frac{3}{2}t^2e^{2t}$

$y = y_h + y_p = (C_1 + C_2t)e^{2t} + \frac{3}{2}t^2e^{2t}$

Husk: $y'' + a(t)y' + b(t)y = h(t)$ (lineær andre ordens)

Så ofte $y = y_h + y_p$, men vi kan kun bruke karakteristisk likn for y_h hvis $a(t) = a$, $b(t) = b$ er konstanter

Eks: $y'' - 4y' + 3y = 6$, $y(0) = 2$, $y'(0) = 2e$

$y = y_h + y_p = \underline{c_1 e^t + c_2 e^{3t} + 2}$ $\quad y' = \underline{c_1 e^t + 3c_2 e^{3t}}$

y_h: $r^2 - 4r + 3 = 0$

$(r-1)(r-3) = 0$ $\quad y_h = \underline{c_1 e^t + c_2 e^{3t}}$

y_p: $y'' - 4y' + 3y = 6$ $\quad y = A$

$3A = 6$ $\quad y' = 0$ $\quad y'' = 0$

$A = 2$ $\quad y_p = \underline{2}$

$y(0) = c_1 e^0 + c_2 e^0 + 2 = 2$

$c_1 + c_2 + 2 = 2$

$c_1 + c_2 = 0$

$\underline{c_2 = -c_1}$

$y'(0) = c_1 e^0 + 3c_2 e^0 = 2e$

$c_1 + 3c_2 = 2e$

$c_1 + 3(-c_1) = 2e$

$\frac{-2c_1}{-2} = \frac{2e}{-2}$

$c_1 = -e$

$\leftarrow$ $c_2 = e$

$y = -e \cdot e^t + e \cdot e^{3t} + 2$

$= -e^{t+1} + e^{3t+1} + 2$

$y = \underline{\underline{e^{3t+1} - e^{t+1} + 2}}$

Eks.: $y'' + 6y' - 7y = 0$

$r^2 + 6r - 7 = 0$ karakteristisk lik.

$(r+7)(r-1) = 0$

$y = c_1 e^{-7t} + c_2 e^{t}$

$y_1 = y(t)$
$y_2 = y'(t)$

$y_1' = y'(t) = y_2$
$y_2' = y''(t) = -6y' + 7y = -6y_2 + 7y_1$

$y_1' = y_2$
$y_2' = 7y_1 - 6y_2$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 7 & -6 \end{pmatrix}}_{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 7 & -6-\lambda \end{vmatrix} = -\lambda(-6-\lambda) - 7 = 0$$

$$\boxed{\lambda^2 + 6\lambda - 7 = 0}$$