	Emne	Kompendium
1	Repetisjon og oppgaveregning	
2	Variasjonsregning	[DA] 14

Rep. forelesning: 7. mai

Oppgaver for Forelesning 14	
Oppgaver fra arbeidsboken	[DA] 14.1 - 14.13

① Repetisjon: Andre ordens lineære diff.lign.

$y'' + a(t)\cdot y' + b(t)\cdot y = h(t)$ andre ordens lineær

Metode: Superposisjon $y = y_h + y_p$

y_h: $y'' + ay' + by = 0$ $a = a(t)$, $b = b(t)$ konst.

$\Rightarrow$ kan bruke karakteristisk likning.

$r^2 + ar + b = 0$

$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$

$a^2 - 4b > 0$: $r_1 \neq r_2$ $\quad y_h = C_1 \cdot e^{r_1 t} + C_2 \cdot e^{r_2 t}$

$a^2 - 4b = 0$: $r_1 = r_2 = -a/2$ $\quad y_h = C_1 e^{r_1 t} + C_2 \cdot te^{r_1 t} = (C_1 + C_2 t) e^{r_1 t}$

y_p: $y'' + ay' + by = h(t)$

Velg $y = y(t)$, en funksjon av samme form som $h(t)$, $h'(t)$, $h''(t)$ med en eller flere parametre

Bestem parametrene ved å sette inn i diff.ligningen

(2) Variasjonsregning

En **funksjonal** er en funksjon der input og output er funksjoner (eller output er et tall)

Eks: $J(y) = y'' + y' - 2y$ differensialoperator

$J(t) = 1 - 2t$

$J(e^t) = e^t + e^t - 2e^t = 0$

Eks: $J(y) = \int_0^1 y^2 + |y'|^2 \, dt$

$J(t) = \int_0^1 t^2 + 1 \, dt = \left[\frac{1}{3}t^3 + t\right]_0^1 = 4/3$

Eks: Variasjonsproblem

$$\min \underbrace{\int_0^1 4ty + |y'|^2 \, dt}_{J(y)} \quad \text{når} \quad \begin{matrix} y(0) = 0 \\ y(1) = 1 \end{matrix}$$

Euler-likningen:

$$\max/\min \int_0^T F(t, y, y') \, dt \quad \text{når} \quad \begin{cases} y(0) = y_0 \\ y(T) = y_T \end{cases}$$

generelt variasjonsproblem

Hvis $y = y(t)$ er max/min, så må y oppfylle Euler-likningen:

$$\boxed{F'_y - \frac{d}{dt}(F'_{y'}) = 0}$$

Eks: $F = 4ty + \dot{y}^2$

$F'_y = 4t$

$F'_{\dot{y}} = 2\dot{y} \quad \frac{d}{dt}(F'_{\dot{y}}) = \frac{d}{dt}(2\dot{y}) = 2\ddot{y}$

$\dot{y} = y'$ Newton's notasjon

$\ddot{y} = y''$

Euler: $4t - 2\ddot{y} = 0$

$2y'' = 4t \quad | :2$

$y'' = 2t$

$y' = \int 2t\,dt = t^2 + C_1$

$y = \int t^2 + C_1\,dt$

$y = \frac{1}{3}t^3 + C_1 t + C_2$

$y(0) = 0$: $0 = \frac{1}{3}0^3 + C_1 \cdot 0 + C_2 \Rightarrow C_2 = 0$

$y(1) = 1$: $1 = \frac{1}{3}\cdot 1^3 + C_1 \cdot 1$ $\quad C_1 = 2/3$

Funksjoner som oppfyller Euler-Lokn. + initialbetingelser

$y = \frac{1}{3}t^3 + \frac{2}{3}t \qquad y' = t^2 + 2/3$

$$J(y) = \int_0^1 4ty + \dot{y}^2\,dt = \int_0^1 4t\left(\tfrac{1}{3}t^3 + \tfrac{2}{3}t\right) + \left(t^2 + 2/3\right)^2 dt$$

$$= \int_0^1 \tfrac{4}{3}t^4 + \tfrac{8}{3}t^2 + t^4 + \tfrac{4}{3}t^2 + \tfrac{4}{9}\,dt$$

$$= \left[\tfrac{7}{3}\cdot\tfrac{1}{5}t^5 + 4\cdot\tfrac{1}{3}t^3 + \tfrac{4}{9}t\right]_0^1 = \tfrac{7}{15} + \tfrac{4}{3} + \tfrac{4}{9}$$

$$= \frac{7\cdot 3 + 4\cdot 15 + 4\cdot 5}{45} = \frac{101}{45}$$

Def: $F(t,y,\dot{y})$ er konvex i $(y,\dot{y})$ hvis

følgende betingelser er oppfyllt:

$F''_{yy}\cdot F''_{\dot{y}\dot{y}} - (F''_{y\dot{y}})^2 \geq 0$ og $F''_{yy} \geq 0$, $F''_{\dot{y}\dot{y}} \geq 0$

og konkav i $(y,\dot{y})$ hvis

$F''_{yy}\cdot F''_{\dot{y}\dot{y}} - (F''_{y\dot{y}})^2 \geq 0$ og $F''_{yy} \leq 0$, $F''_{\dot{y}\dot{y}} \leq 0$

Eks: $F = 4ty + \dot{y}^2$

$F'_y = 4t$

$F'_{\dot{y}} = 2\dot{y}$ ..

$F''_{yy} = 0 \quad F''_{y\dot{y}} = 0$

$F''_{\dot{y}y} = 0 \quad F''_{\dot{y}\dot{y}} = 2$

$H(F) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ $\det = 0 \geq 0$ og $0, 2 \geq 0$ ok.

$\Rightarrow$ F konvex i $(y, \dot{y})$ ($\Leftrightarrow$ $H(F)$ positiv semidefin.)

Resultat:

Hvis $y = y(t)$ er løsning av Euler-likningen og initialbetingelsene, så har vi:

i) Hvis F er konvekst i $(y, \dot{y})$, så er y min

ii) Hvis F er konkav i $(y, \dot{y})$, så er y max.

Eks: Eksamen V'13, Oppg. 6

$\max/\min \int_0^3 \ln(4y - \dot{y})\, dt$ når $y(0) = 3$, $y(3) = -9e^{12}$

$F = \ln(u)$, der $u = 4y - \dot{y}$

$F'_y = \frac{1}{u} \cdot u'_y = \frac{4}{4y - \dot{y}}$

$F'_{\dot{y}} = \frac{1}{u} \cdot u'_{\dot{y}} = \frac{-1}{4y - \dot{y}}$

Euler: $\frac{4}{u} - \frac{d}{dt}\left(\frac{-1}{u}\right) = 0$

$\frac{4}{u} + \frac{d}{dt}(u^{-1}) = 0$

$\frac{4}{u} + (-1)u^{-2} \cdot u'_t = 0$

$u \cdot \frac{4}{u} - \frac{4\dot{y} - \ddot{y}}{u^2} = 0$

Euler: $\dfrac{4(4y-\dot y)-(4\dot y-\ddot y)}{(4y-\dot y)^2}=0$

$\ddot y-8\dot y+16y=0$

$r^2-8r+16=0$

$(r-4)^2=0$

$r_1=r_2=4$

$y=y_h=(c_1+c_2t)e^{4t}$

$y(0)=3$: $(c_1+c_2\cdot 0)e^0=3$

$c_1=3$

$y(3)=-9e^{12}$:

$(3+c_2\cdot 3)e^{12}=-9e^{12}$

$3c_2=-9-3=-12$

$c_2=-4$

Kandidat for max/min:

$y=(3-4t)e^{4t}$

Er F konvels/konkav i $(y,\dot y)$?

$F''_{yy}=\left(\frac{4}{u}\right)'_y=4(-1)u^{-2}\cdot u'_y=-\frac{16}{u^2}$

$F''_{y\dot y}=\left(\frac{4}{u}\right)'_{\dot y}=4(-1)u^{-2}\cdot u'_{\dot y}=\frac{4}{u^2}$

$F''_{\dot y\dot y}=\left(\frac{-1}{u}\right)'_{\dot y}=-1\cdot(-1)u^{-2}\cdot u'_{\dot y}=-\frac{1}{u^2}$

$H(F)=\begin{pmatrix}-\frac{16}{u^2} & \frac{4}{u^2}\\ \frac{4}{u^2} & -\frac{1}{u^2}\end{pmatrix}$

$\det H(F)=\frac{16}{u^4}-\frac{16}{u^4}=0$ ✓

$y^*=(3-4t)e^{4t}$ ⇐ F konkav ⇐ $-\frac{16}{u^2},\ -\frac{1}{u^2}\le 0$

er max

$\dot y=-4e^{4t}+(3-4t)\cdot 4e^{4t}$

$=e^{4t}(8-12t)$

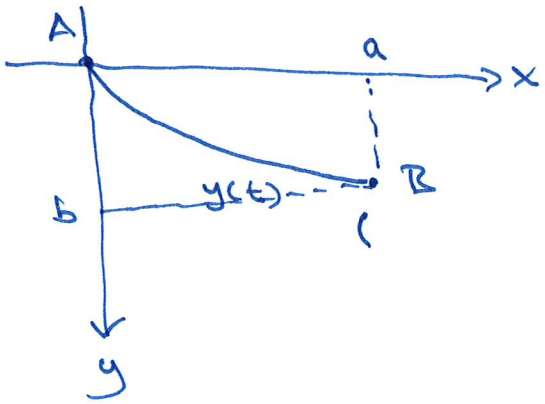
$=-\frac{16}{(4y-\dot y)^2}$

Max-verdi av fullspralen:

$\int_0^3 \ln\left(4(3-4t)e^{4t}-(8-12t)e^{4t}\right)dt=\int_0^3 \ln\left((4-4t)e^{4t}\right)dt$

$=\int_0^3 \ln(4-4t)+4t\,dt=\dots$

Klassisk problem



$$\min \int_0^a \sqrt{\frac{1+y'^2}{2gy}}\,dx$$

$y(0)=0$ når $y(a)=b$

Eks: (makro)

$$\max \int_0^T u(c)\cdot e^{-rt}\,dt = \int_0^T u(f(k)-k')\,e^{-rt}\,dt$$

u : nyttefunksjon
C : forbruk

Y = netto nasjonalprodukt
k = kapitalmengde

$Y = C + K'$

$Y = f(k)$

Euler-ligningen:

$$\max/\min\ J(y) = \int_0^T F(t,y,y')\,dt$$

Euler-likn: "$J'=0$" $\Leftrightarrow$ $\boxed{F_y' - \frac{d}{dt}(F_{y'}')=0}$

Konkav/konveks: Når er en symmetrisk 2×2-matrise positiv semidefinitt / positiv definitt?

$$A=\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

pos. def. $\Leftrightarrow \lambda_1,\lambda_2>0$
pos. semidef. $\Leftrightarrow \lambda_1,\lambda_2\geq 0$

$$\lambda^2-(a+c)\lambda+(ac-b^2)=0$$

$$\lambda=\frac{a+c\pm\sqrt{(a+c)^2-4(ac-b^2)}}{2}$$

$$\Delta=(a+c)^2-4(ac-b^2)$$
$$=a^2+2ac+c^2-4ac+4b^2$$
$$=a^2-2ac+c^2+4b^2=(a-c)^2+4b^2$$

$\Rightarrow$ to egenverdier λ_1,λ_2

$ac-b^2<0 \Rightarrow \sqrt{\Delta} > a+c \Rightarrow$ minst en negativ egenverdi

Konkl:

A pos. def. $\Leftrightarrow$ $ac-b^2>0$, $a>0$

A pos. semidef. $\Leftrightarrow$ $ac-b^2\geq 0$, $a\geq 0$, $c\geq 0$

Eks: max/min $\int_0^1 y^2+\dot{y}^2+t^2\,dt$ når $y(0)=0$, $y(1)=1$

$F'_y = 2y$

$F'_{\dot{y}} = 2\dot{y}$ $\quad \frac{d}{dt}(2\dot{y}) = 2\ddot{y}$

Euler: $2y-2\ddot{y}=0 \quad |\cdot -2$

$\ddot{y}-y=0$

$r^2-1=0$

$r=\pm 1$ $\quad y = \underline{c_1e^t+c_2e^{-t}}$

$y(0)=0$: $c_1e^0+c_2e^{-0}=0$

$c_1+c_2=0$ $\quad \underline{c_2=-c_1}$

$y(1)=1$: $c_1e^1+c_2e^{-1}=1$

$c_1\cdot e - c_1\cdot\frac{1}{e}=1$

$c_1(e-\frac{1}{e})=1$

$c_1 = \frac{1}{e-\frac{1}{e}}\cdot e = \underline{\frac{e}{e^2-1}}$

$c_2 = \underline{\frac{-e}{e^2-1}}$

$y(t) = \frac{e}{e^2-1}e^t - \frac{e}{e^2-1}e^{-t}$

$= \underline{\underline{\frac{e}{e^2-1}(e^t-e^{-t})}}$

$F''_{yy}=2 \quad F''_{y\dot{y}}=0$

$F''_{\dot{y}y}=0 \quad F''_{\dot{y}\dot{y}}=2$

$H(F)=\begin{pmatrix}2&0\\0&2\end{pmatrix}$ F konvekss

$\det = 4\geq 0$

$2,2\geq 0$

$y^* = \frac{e}{e^2-1}(e^t-e^{-t})$ er min

Eks: max/min $\int_0^1 (4ty - \dot{y}^2) \cdot e^{-rt}\, dt$ når $y(0)=0$, $y(1)=1$

(r er en gitt konstant)

Euler: $F = (4ty - \dot{y}^2)\, e^{-rt}$

$F'_y = 4t \cdot e^{-rt}$

$F'_{\dot{y}} = -2\dot{y} \cdot e^{-rt}$ $\quad \frac{d}{dt}(-2\dot{y}\, e^{-rt}) = -2\ddot{y}\, e^{-rt} - 2\dot{y}\, e^{-rt} \cdot (-r)$

$= (-2\ddot{y} + 2r\dot{y})\, e^{-rt}$

$4t \cdot e^{-rt} - (-2\ddot{y} + 2r\dot{y})\, e^{-rt} = 0$

$2y'' - 2ry' = -4t \quad | :2$ } $y = y_h + y_p$

$y'' - ry' = -2t$

y_h: $\lambda^2 - r\lambda = 0$

$\lambda(\lambda - r) = 0$

$\lambda = 0, \ \lambda = r$ } $y_h = C_1 e^{0\cdot t} + C_2 e^{rt} = C_1 + C_2 e^{rt}$

y_p: $y'' - ry' = -2t$

$h = -2t$, $h' = -2$, $h'' = 0$

$y = At + B$, $y' = A$, $y'' = 0$ (crossed out)

$y = At^2 + Bt$, $y' = 2At + B$, $y'' = 2A$

$0 - r \cdot A = -2t$

$2A - r(2At + B) = -2t$

$(-2rA)t + (2A - rB) = -2t + 0$

$-2rA = -2 \quad A = \frac{-2}{-2r} = \frac{1}{r}$

$2A - rB = 0 \quad \frac{2}{r} - rB = 0$

$rB = \frac{2}{r} \quad B = \frac{2}{r^2}$

} $y_p = \frac{1}{r}t^2 + \frac{2}{r^2}t$

$$y = c_1 + c_2 e^{rt} + \frac{1}{r} t^2 + \frac{2}{r^2} t$$

$$\left.\begin{matrix} y(0) = 0 \\ y(1) = 1 \end{matrix}\right\} \rightarrow \begin{matrix} c_1 = \dots \\ c_2 = \dots \end{matrix}$$

Konvekst / konkav:

$$F'_y = 4te^{-rt}$$

$$F'_y = -2y e^{-rt}$$

$$H(F) = \begin{pmatrix} 0 & 0 \\ 0 & -2e^{-rt} \end{pmatrix} \qquad \det = 0,\ 0,\ -2e^{-rt} \leq 0$$

F konkav $\Rightarrow y^*$ er max

$y(0) = 0$: $c_1 + c_2 = 0$, $c_2 = -c_1$

$y(1) = 1$: $c_1 - c_1 e^r + \frac{1}{r} + \frac{2}{r^2} = 1$

$$c_1 = \frac{1 - 1/r - 2/r^2}{1 - e^r}$$

$$c_1(1 - e^r) = -\frac{1}{r} - \frac{2}{r^2} + 1$$

$$c_2 = -\frac{1 - 1/r - 2/r^2}{1 - e^r}$$

$$y^* = \frac{1 - 1/r - 2/r^2}{1 - e^r}\left(1 - e^{rt}\right) + \frac{1}{r} t^2 + \frac{2}{r^2} t$$