Emne

1 Om eksamen
2 Gjennomgang av Eksamen 05/2025 + 05/2024 2d

① Formelark: som vist tidligere
Eksamen: 5t 15 deloppgaver
Husk: Kursevaluering

② Eksamen 05/2025

1. a) $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 = 0$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 2 & -1 & 12 & 8 \\ 4 & 3 & 14 & 7 \end{pmatrix} \begin{matrix} \\ \cdot(-2) \\ \cdot(-4) \end{matrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -5 & 10 & 10 \\ 0 & -5 & 10 & 11 \end{pmatrix} \cdot(-1)$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -5 & 10 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$x_1 + 2x_2 + x_3 - x_4 = 0$
$-5x_2 + 10x_3 + 10x_4 = 0$
$+x_4 = 0$

$x_3 = t$ fri variabel/parameter

$x_4 = 0$ $\quad -5x_2 = -10t - 10 \cdot 0 = -10t$

$x_2 = 2t$

$x_1 = -2(2t) - (t) + (0) = -5t$

Løsn: $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -5t \\ 2t \\ t \\ 0 \end{pmatrix} = t \cdot \begin{pmatrix} -5 \\ -2 \\ 1 \\ 0 \end{pmatrix}$ (t parameter)

$-5v_1 - 2v_2 + v_3 = 0 \Rightarrow v_3 = 5v_1 + 2v_2 + 0 \cdot v_4$
Nei, de er lin. avhengige siden $v_3 = 5v_1 + 2v_2$

b) $W = \text{Span}(v_1, v_2, v_4)$

v_1, v_2, v_4 lin uavh fra pivotpos. i trappeformen i a) $\Rightarrow$ Basis for W: $\{v_1, v_2, v_4\}$

$V = \text{Span}(v_1, v_2, v_3)$

$= \text{Span}(v_1, v_2)$ ~~$\neq$~~ $W = \text{Span}(v_1, v_2, v_4)$

fordi v_3 er lin.komb. av v_1, v_2

Ja, v_4 er i W men ikke i V (for eks!),

og de vektorene i W som ikke ligger i V har formen: $x_1 v_1 + x_2 v_2 + x_4 v_4$ med $x_4 \neq 0$

2. $A = \begin{pmatrix} 4 & 1 & 4 \\ 1 & 4 & 4 \\ 4 & 4 & 9 \end{pmatrix}$

a) $\lambda = 1$: $A - I = \begin{pmatrix} 3 & 1 & 4 \\ 1 & 3 & 4 \\ 4 & 4 & 8 \end{pmatrix}$ $R(1) + R(2) = R(3) \Rightarrow |A - I| = 0$

$\Rightarrow \lambda = 1$ er en egenverdi

$\lambda = 2$, $\lambda = 3$ tilsvarende

Alt: $\begin{vmatrix} 4-\lambda & 1 & 4 \\ 1 & 4-\lambda & 4 \\ 4 & 4 & 9-\lambda \end{vmatrix} = 0$

$(4-\lambda)\cdot((4-\lambda)(9-\lambda)-16) - 1\left((9-\lambda)-16\right) + 4\left(4 - 4(4-\lambda)\right) = 0$

$(4-\lambda)\cdot(\lambda^2 - 13\lambda + 20) + \lambda + 7 + 16\lambda - 48 = 0$

$\lambda = 1$: $3 \cdot 8 + 8 + (-32) = 0$ $\lambda = 1$ egenverdi

$\lambda = 2$: $2 \cdot (-2) + 9 - 16 = -11$ $\lambda = 2$ ikke egenverdi

$\lambda = 3$: $1 \cdot (-10) + 10 + 0 = 0$ $\lambda = 3$ egenverdi

$\lambda_1=1$, $\lambda_2=3$ er egenverdier for A

$\lambda_1+\lambda_2+\lambda_3 = tr(A)$

$1+3+\lambda_3 = 17 \quad \lambda_3 = 13$

Egenverdier:

$\lambda_1=1, \lambda_2=3, \lambda_3=13$

Alt $\lambda_1\cdot\lambda_2\cdot\lambda_3 = \det(A)$

$1\cdot 3\cdot\lambda_3 = 39$

b) $\lambda=1$: $\begin{pmatrix}3&1&4\\1&3&4\\4&4&8\end{pmatrix} \to \begin{pmatrix}1&3&4\\3&1&4\\4&4&8\end{pmatrix} \begin{matrix}\\ -3\\ -4\end{matrix} \to \begin{pmatrix}1&3&4\\0&-8&-8\\0&-8&-8\end{pmatrix}$

$x+3y+4z=0$

$-8y-8z=0$

z fri

$x = -3(-z)+4z = -z$

$y=-z$

$\underline{x} = z\begin{pmatrix}-1\\-1\\1\end{pmatrix}$ $\quad \underline{v}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\-1\\1\end{pmatrix}$

$\lambda=3$: $\begin{pmatrix}1&1&4\\1&1&4\\4&4&6\end{pmatrix} \begin{matrix}\\ -1\\ -4\end{matrix} \to \begin{pmatrix}1&1&4\\0&0&-10\\0&0&0\end{pmatrix}$

$x+y+4z=0$

$-10z=0$

y fri

$z=0 \quad x=-y$

$\underline{x} = \begin{pmatrix}-y\\y\\0\end{pmatrix} = y\cdot\begin{pmatrix}-1\\1\\0\end{pmatrix}$ $\quad \underline{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}$

$\lambda=13$: $\begin{pmatrix}-9&1&4\\1&-9&4\\4&4&-4\end{pmatrix} \to \begin{pmatrix}1&-9&4\\-9&1&4\\4&4&-4\end{pmatrix} \begin{matrix}\\ +9\\ -4\end{matrix} \to \begin{pmatrix}1&-9&4\\0&-80&40\\0&40&-20\end{pmatrix} \begin{matrix}\\ \\ +1/2\end{matrix}$

$x-9y+4z=0$

$-80y+40z=0$

z fri

$x = 9(z/2)-4z = z/2$

$y = z/2$

$\underline{x} = \frac{z}{2}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}$ $\quad \underline{v}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\2\end{pmatrix}$

$$P = \begin{pmatrix}-1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6}\\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6}\\ 1/\sqrt{3} & 0 & 2/\sqrt{6}\end{pmatrix}$$

siden $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ er ortonormal basis av egenvektorer

c) $A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A) = \frac{1}{1\cdot 3\cdot 13}\begin{pmatrix}20 & 7 & -12\\ 7 & 20 & -12\\ -12 & -12 & 15\end{pmatrix}^T = \frac{1}{39}\begin{pmatrix}20 & 7 & -12\\ 7 & 20 & -12\\ -12 & -12 & 15\end{pmatrix}$

Egenverdier for A^{-1}: $\frac{1}{1} = 1$, $\frac{1}{3}$, $\frac{1}{13} > 0$

fordi: Hvis λ er en egenverdi for A,
så er $A\cdot\underline{v} = \lambda\cdot\underline{v}$ har ikke-trivielle løsn. for $\underline{v}$

$$A^{-1}A\underline{v} = A^{-1}\lambda\underline{v}$$
$$\underline{v} = \lambda\cdot A^{-1}\underline{v}$$
$$\frac{1}{\lambda}\cdot\underline{v} = A^{-1}\underline{v} \quad \text{— " —}$$

$\Rightarrow$ $\frac{1}{\lambda}$ egenverdi for A^{-1}

$\Rightarrow$ A^{-1} er pos. def

d) $f(\underline{x}) = \underline{x}^T A\underline{x} + B\underline{x} + 22$

$$A = \begin{pmatrix}4 & 1 & 4\\ 1 & 4 & 4\\ 4 & 4 & 9\end{pmatrix} \qquad B = (0\ 6\ 10)$$

$\lambda = 1, 3, 13$

$\Rightarrow$ A pos. def.

$f'(\underline{x}) = 2A\underline{x} + B^T = \underline{0}$

$$A\underline{x} = -\tfrac{1}{2}B^T \qquad \left(\begin{array}{ccc|c}4 & 1 & 4 & 0\\ 1 & 4 & 4 & -3\\ 4 & 4 & 9 & -5\end{array}\right)$$

$\underline{x} = A^{-1}\cdot\left(-\frac{1}{2}B^T\right)$

$$= \frac{1}{39}\begin{pmatrix}20 & 7 & -12\\ 7 & 20 & -12\\ -12 & -12 & 15\end{pmatrix}\cdot\begin{pmatrix}0\\ -3\\ -5\end{pmatrix} = \frac{1}{39}\begin{pmatrix}39\\ 0\\ -39\end{pmatrix} = \begin{pmatrix}1\\ 0\\ -1\end{pmatrix}$$

stasj. pkt min pkt pga A pos-def.

d) $52x_1^2 + 73x_2^2 - 72x_1x_2 = 2500$

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$: $\underline{x}^T A \underline{x} = 2500$ $\quad A = \begin{pmatrix} 52 & -36 \\ -36 & 73 \end{pmatrix}$

$|A - \lambda I| = \lambda^2 - 125\lambda + 2500 = 0$

$\lambda = 25, \lambda = 100$

$P = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}$ $\quad D = \begin{pmatrix} 25 & 0 \\ 0 & 100 \end{pmatrix}$

$\lambda = 25$: $\begin{pmatrix} 27 & -36 \\ -36 & 48 \end{pmatrix}$ $\cdot 4/3$

$\rightarrow \begin{pmatrix} 27 & -36 \\ 0 & 0 \end{pmatrix}$ $\quad 27x - 36y = 0$, y fri

$x = \frac{36}{27}y = \frac{4}{3}y$

$\underline{v}_1 = \frac{y}{3}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$\underline{x}^T A \underline{x} = 2500$ $\quad \underline{x} = P\underline{y}$

$(P\underline{y})^T A (P\underline{y}) = 2500$

$\underline{y}^T \underbrace{P^T A P}_{D} \underline{y} = 2500$

$25y_1^2 + 100y_2^2 = 2500 \quad | : 2500$

$$\frac{y_1^2}{100} + \frac{y_2^2}{25} = 1$$

$\lambda = 100$: ~~$\begin{pmatrix} 27 & -36 \\ -36 & -52 \end{pmatrix}$~~

$\begin{pmatrix} -48 & -36 \\ -36 & -27 \end{pmatrix}$ $\cdot -3/4 \rightarrow \begin{pmatrix} -48 & -36 \\ 0 & 0 \end{pmatrix}$

$-48x - 36y = 0 \quad x = -\frac{3}{4}y$

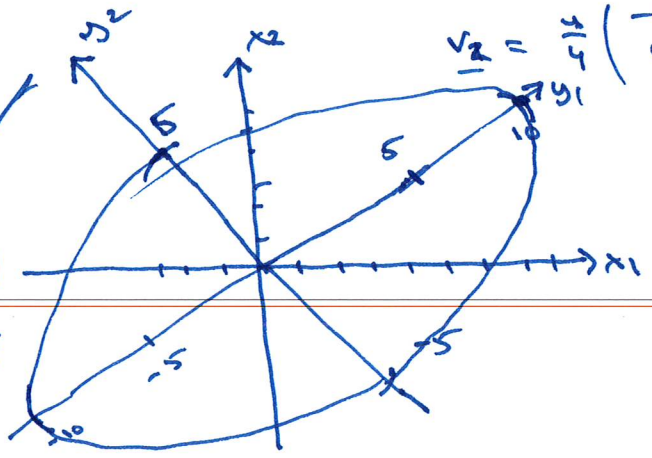
$\underline{v}_2 = \frac{y}{4}\begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$\underline{y} = P^T \cdot \underline{x} = P^T \cdot \underline{x}$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$y_1 = \frac{4x_1 + 3x_2}{5}$

$y_2 = \frac{-3x_1 + 4x_2}{5}$



$\underline{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $\underline{x} = P \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$

$\underline{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$: $\underline{x} = P \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3/5 \\ 4/5 \end{pmatrix}$

3. a) i) $f(x,y) \geq 0$ for alle x,y $\leftarrow$ $x^2+xy+y^2 = \underline{x}^T A \underline{x} \geq 0$ da

ii) $\int_{-1}^{1}\int_{-3}^{3} f(x,y)\,dy\,dx = 1$ $\qquad A = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$

$\lambda^2 - 2\lambda + 3/4 = 0$

$(\lambda-1)^2 = 1/4$

$\lambda - 1 = \pm 1/2$

$\lambda = 1 \pm 1/2 > 0$

A pos. def.

$$\int_{-1}^{1}\int_{-3}^{3} \frac{1}{40}(x^2+xy+y^2)\,dy\,dx$$

$$= \frac{1}{40}\int_{-1}^{1} \left[x^2y + x\cdot\tfrac{1}{2}y^2 + \tfrac{1}{3}y^3\right]_{-3}^{3} dx$$

$$= \frac{1}{40}\int_{-1}^{1} \left(3x^2 + \tfrac{3}{2}x + 9\right) - \left(-3x^2 + \tfrac{3}{2}x - 9\right) dx$$

$$= \frac{1}{40}\left[6\cdot\tfrac{1}{3}x^3 + 18x\right]_{-1}^{1} = \frac{1}{40}\left((2+18)-(-2-18)\right) = 1 \text{ da.}$$

b) $f_x(x) = \int_{-3}^{3} \frac{1}{40}(x^2+xy+y^2)\,dy = \frac{1}{40}(6x^2+18)$

$$E(X) = \int_{-1}^{1} x\cdot\frac{1}{40}(6x^2+18)\,dx = \frac{1}{40}\left[6\cdot\tfrac{1}{4}x^4 + 18\cdot\tfrac{1}{2}x^2\right]_{-1}^{1}$$

$$= 0$$

c) X og Y uavh. $\Leftrightarrow$ $f_x(x)\cdot f_y(y) = f(x,y)$ $\leftarrow$ kravet

$$\frac{1}{40}(6x^2+18)\cdot\frac{1}{40}\left(\tfrac{2}{3}+2y^2\right) \neq \frac{1}{40}(x^2+xy+y^2)$$

$x=0, y=0$:

$\frac{18}{40}\cdot\frac{2/3}{40} \neq 0$

$\Rightarrow$ X, Y ikke uavhengige

$$f_y(y) = \int_{-1}^{1} \frac{1}{40}(x^2+xy+y^2)\,dx = \frac{1}{40}\left[\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2y + xy^2\right]_{-1}^{1}$$

$$= \frac{1}{40}\left(\left(\tfrac{1}{3}+\tfrac{1}{2}y+y^2\right) - \left(-\tfrac{1}{3}+\tfrac{1}{2}y-y^2\right)\right) = \frac{1}{40}\left(\tfrac{2}{3}+2y^2\right)$$

d) $f_{X|Y=1}(x) = \frac{f(x,1)}{f_Y(1)} = \frac{\frac{1}{40}(x^2+x+1)}{\frac{1}{40}(2/3+2)} = \frac{x^2+x+1}{8/3}$

$= \frac{3}{8}(x^2+x+1)$

$E[X|Y=1] = \int_{-1}^{1} x \cdot \frac{3}{8}(x^2+x+1)\,dx$

$= \frac{3}{8}\left[\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2\right]_{-1}^{1} = 3/8\left(\frac{1}{3}\cdot 1^3 - \frac{1}{3}(-1)^3\right)$

$= 3/8 \cdot 2/3 = 2/8 = 1/4$

4. a) $y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r_1 = r_2 = 2 \Rightarrow y = C_1 \cdot e^{2t} + C_2\, t \cdot e^{2t}$

b) $y' = e^{t-y} = e^t \cdot e^{-y} = e^t \cdot \frac{1}{e^y} \quad | \cdot e^y$

$e^y \cdot y' = e^t \qquad \int e^y y'\,dt = \int e^t\,dt$

$e^y = \int e^y\,dy = e^t + C \quad | \ln(\cdot)$

$y = \ln(e^t + C)$

c) $y' = \frac{t+y}{t} = 1 + \frac{y}{t} = 1 + \frac{1}{t}\cdot y$

$y' - \frac{1}{t}\cdot y = 1 \cdot \frac{1}{t}$

$\frac{1}{t}y' - \frac{1}{t^2}y = \frac{1}{t}$

$\left(\frac{1}{t}\cdot y\right)' = \frac{1}{t}$

$\frac{1}{t}\cdot y = \int \frac{1}{t}\,dt$

$\frac{1}{t}\cdot y = \ln t + C \quad | \cdot t$

$y = t\ln t + Ct \qquad t>0$

$y' + a(t)\cdot y = b(t)$

Int. faktor:

$\int a(t)\,dt = \int -\frac{1}{t}\,dt = -\ln|t| + C$

$u = e^{-\ln t} = \frac{1}{e^{\ln t}} = \frac{1}{t}$

5. max $\int_0^1 \ln(y-u)\,dt$ når $\begin{cases} y' = y+u \\ y(0)=1 \\ y(1)=0 \end{cases}$

a) $H = \ln(y-u) + p\cdot(y+u)$ ← Hamilton normal løsn. ($p_0=1$)

$H'_u = \frac{1}{y-u}\cdot(-1) + p = 0$

$H'_y = \frac{1}{y-u}\cdot 1 + p = -p'$

II

$-\frac{1}{y-u} + p = 0$

$\frac{1}{y-u} + p = -p'$

$y' = y+u$

$\rightarrow p = \frac{1}{y-u} = (y-u)^{-1}$

$\frac{1}{y-u}\cdot 2 = -(-1)(y-u)^{-2}\cdot(y'-u')$

$= \frac{1}{(y-u)^2}\cdot(y'-u') \quad |\cdot(y-u)^2$

$2(y-u) = y' - u'$

$2(y-(y'-y)) = y' - (y''-y')$

$u = y' - y$

$u' = y'' - y'$

$y'' - 4y' + 4y = 0 \qquad y = c_1 e^{2t} + c_2 t e^{2t} = (c_1 + c_2 t)\cdot e^{2t}$

$y(0)=1 : \; c_1 e^0 = 1 \quad c_1 = 1$

$y(1)=0 \quad (1+c_2)e^2 = 0 \quad c_2 = -1$

$\Rightarrow y = (1-t)e^{2t}$

$H_y' = \frac{1}{y-u} \qquad H_u' = \frac{-1}{y-u}$

$H_{yy}'' = -\frac{1}{(y-u)^2} \qquad H_{yu}'' = \frac{1}{(y-u)^2}$

$H_{uy}'' = \frac{1}{(y-u)^2} \qquad H_{uu}'' = -\frac{1}{(y-u)^2}$

$\det = \frac{1}{(y-u)^4} - \frac{1}{(y-u)^4} = 0$

$-\frac{1}{(y-u)^2} < 0$

$\Rightarrow$ H konkav

$y^* = (1-t)e^{2t}$

gir maks.

b) $y' = y + u \Rightarrow u = y' - y \qquad y - u = y - (y' - y) = 2y - y'$

$\max \int_0^1 \underbrace{\ln(2y-y')}_{F}\, dt$ når $y(0)=1$, $y(1)=0$

$F_y' = \frac{1}{2y-y'}\cdot 2 = \frac{2}{2y-y'}$

$F_{y'}' = \frac{1}{2y-y'}\cdot(-1) = \frac{-1}{2y-y'} \Rightarrow \frac{d}{dt}\left(F_{y'}'\right) = 1\cdot(2y-y')^{-2}\cdot(2y'-y'') - 1\cdot(2y-y')^{-1}\cdot y''$

$= \frac{2y'-y''}{(2y-y')^2}$

Euler: $\frac{2}{2y-y'} - \frac{2y'-y''}{(2y-y')^2} = 0 \quad | \cdot (2y-y')^2$

$$2 \cdot (2y-y') - (2y'-y'') = 0$$

$$y'' - 4y' + 4y = 0 \Rightarrow y^* = (1-t)e^{2t}$$

$\Leftarrow$ oppg (a)

$$F'_y = \frac{2}{2y-y'}$$

$$F'_{y'} = \frac{-1}{2y-y'}$$

$$F''_{yy} = \frac{-2 \cdot 2}{(2y-y')^2} \qquad F''_{yy'} = \frac{-2 \cdot (-1)}{(2y-y')^2}$$

$$F''_{y'y} = \frac{2}{(2y-y')^2} \qquad F''_{y'y'} = \frac{-1(-1)\cdot(-1)}{(2y-y')^2}$$

$$= \frac{1}{(2y-y')^2} \cdot \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}$$

F konkav $\Leftarrow$ $\det = \frac{1}{(2y-y')^4}(4-4) = 0$

$\Downarrow$

$y^* = (1-t)e^{2t}$ er maks.

$$\frac{-4}{(2y-y')^2} \cdot \frac{-1}{(2y-y')^2} < 0$$