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① Repetisjon

X og Y simultant fordelte kont. stokastisk variabler

Sannsynlighetstetthet: $f(x,y)$

Eks: Vi kaster to terninger

X = diff. Y = max

x \ y	1	2	3	4	5	6	$p(X=x)$
0	1/36	1/36	1/36	1/36	1/36	1/36	6/36
1	0	2/36	2/36	2/36	2/36	2/36	10/36
2	0	0	2/36	2/36	2/36	2/36	8/36
3	0	0	0	2/36	2/36	2/36	6/36
4	0	0	0	0	2/36	2/36	4/36
5	0	0	0	0	0	2/36	2/36
$p(X=x,Y=y)$							1
$p(Y=y)$ →	1/36	3/36	5/36	7/36	9/36	11/36	

Marginals:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Forventning:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy$$

X og Y er uavhengige hvis $p(X=x, Y=y) = p(X=x) \cdot p(Y=y)$

$x=2, y=3$:

$$\frac{2}{36} \neq \frac{8}{36} \cdot \frac{5}{36}$$

(2) Forventning, varians og kovarians

X, Y simultant fordelte kont. stok. variable med simultan tetthetsfk. $f(x,y)$

Da har vi:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Defn: La $h(X,Y)$ være en funksjon av X og Y

$$E[h(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\cdot f(x,y)\,dy\,dx$$

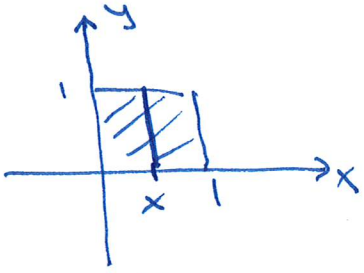
Eks:

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\cdot f(x,y)\,dy\,dx = \int_{-\infty}^{\infty} x\cdot\left(\int_{-\infty}^{\infty} f(x,y)\,dy\right)dx$$

$$= \int_{-\infty}^{\infty} x\cdot f_X(x)\,dx$$

$$E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\cdot f(x,y)\,dy\,dx$$

Eks: $f(x,y) = x+y$, $0 \le x \le 1$, $0 \le y \le 1$



Sannsynlighetstetthet

$$f_X(x) = \int_0^1 x+y\,dy = \left[xy + \tfrac{1}{2}y^2\right]_0^1 = x + 1/2, \quad 0 \le x \le 1$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^1\int_0^1 x+y\,dy\,dx = \int_0^1 f_X(x)\,dx$$

$$= \int_0^1 x + 1/2\,dx = \left[\tfrac{1}{2}x^2 + \tfrac{1}{2}x\right]_0^1 = 1 \quad \text{(ok)}$$

$$E(X) = \int_0^1\int_0^1 x\cdot(x+y)\,dy\,dx = \int_0^1 x \underbrace{\int_0^1 (x+y)\,dy}_{f_X(x)}\,dx$$

$$= \int_0^1 x\,(x+1/2)\,dx = \left[\tfrac{1}{3}x^3 + \tfrac{1}{2}\cdot\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{3}+\tfrac{1}{4}$$

$$= 7/12$$

$$E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\cdot f(x,y)\,dy\,dx = \int_0^1\int_0^1 xy(x+y)\,dy\,dx$$

$$= \int_0^1\int_0^1 x^2y + xy^2\,dy\,dx = \int_0^1 \left[x^2\cdot\tfrac{1}{2}y^2 + x\cdot\tfrac{1}{3}y^3\right]_0^1 dx$$

$$= \int_0^1 \tfrac{1}{2}x^2 + \tfrac{1}{3}x\,dx = \left[\tfrac{1}{2}\cdot\tfrac{1}{3}x^3 + \tfrac{1}{3}\cdot\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{6}+\tfrac{1}{6}$$

$$= 2/6 = 1/3$$

Husk: $E(X) = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx = \mu_X$

$Var(X) = E[(X-\mu_X)^2] = E(X^2) - E(X)^2$

$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy = \mu_Y$

$Var(Y) = E(Y^2) - E(Y)^2$

Def: $Cov(X,Y) = E[(X-\mu_X)\cdot(Y-\mu_Y)]$

$= E[XY - \mu_X Y + X\cdot(-\mu_Y) + \mu_X\mu_Y]$

$= E[XY - \mu_X Y - \mu_Y \cdot X + \mu_X \cdot \mu_Y]$

$= E(XY) - E(\mu_X Y) - E(\mu_Y X) + E(\mu_X\mu_Y)$

$= E(XY) - \mu_X \cdot E(Y) - \mu_Y \cdot E(X) + \mu_X\mu_Y \cdot E(1)$

$= E(XY) - \mu_X \cdot \mu_Y - \mu_Y\mu_X + \mu_X\mu_Y$

$= E(XY) - E(X)\cdot E(Y)$

Resultat: $\boxed{Cov(X,Y) = E(XY) - E(X)\cdot E(Y)}$

Eks: $f(x,y) = x+y$, $0 \le x,y \le 1$

$E(X) = 7/12$ $\quad Var(X) = E(X^2) - E(X)^2 = \frac{11}{144}$

$E(Y) = 7/12$ $\quad Var(Y) = \frac{11}{144}$

$E(XY) = 1/3$ $\quad Cov(X,Y) = E(XY) - E(X)\cdot E(Y)$

$= 1/3 - \left(\frac{7}{12}\right)^2 = \frac{1\cdot 4\cdot 12 - 7^2}{12\cdot 12}$

$= -\frac{1}{144}$

$f_Y(y) = \int_0^1 x+y\,dx$

$= \left[\frac{1}{2}x^2 + xy\right]_0^1 = y + \frac{1}{2}$

$(0 \le y \le 1)$

$E(Y) = \int_0^1 y\cdot(y+1/2)\,dy$

$= \left[\frac{1}{3}y^3 + \frac{1}{2}\cdot\frac{1}{2}y^2\right]_0^1$

$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$

$$E(X^2) = \int_0^1 x^2 \cdot f_X(x)\,dx = \int_0^1 x^3 + \tfrac{1}{2}x^2\,dx = \left[\tfrac{1}{4}x^4 + \tfrac{1}{2}\cdot\tfrac{1}{3}x^3\right]_0^1$$

$x + \tfrac{1}{2}$

$$= \frac{1}{4} + \frac{1}{6} = \frac{3+2}{12} = \underline{5/12}$$

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{5}{12} - \left|\frac{7}{12}\right|^2 = \frac{5\cdot 12 - 7^2}{12^2} = \frac{11}{144}$$

$$\sigma_X = \sqrt{\frac{11}{144}} = \frac{\sqrt{11}}{12} = \sigma_Y$$

Korr. koeff.:

$$\frac{\mathrm{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y} = \frac{\frac{-1}{144}}{\frac{\sqrt{11}}{12}\cdot\frac{\sqrt{11}}{12}} = \frac{\frac{-1}{144}}{\frac{11}{144}} = -\frac{1}{11} \approx -0.09$$

Regneregler for forventningsverdi:

X, Y sim. fordelte $\rightsquigarrow$ $E(h(X,Y))$

i) $E(U \pm V) = E(U) \pm E(V)$ — U, V uttrykk i X og Y

ii) $E(c \cdot U) = c \cdot E(X)$ — U — " — c konst.

iii) $E(C) = C$ — C konst.

Forklaring:

$$E(X + 2Y) = E(X) + E(2Y) = E(X) + 2\cdot E(Y)$$

$$\iint (x+2y)\cdot f(x,y)\,dy\,dx = \iint x\cdot f(x,y)\,dy\,dx + \iint 2y\, f(x,y)\,dy\,dx$$

$$= \iint x\, f(x,y)\,dy\,dx + 2\iint y\, f(x,y)\,dy\,dx$$

$$E(4) = 4\cdot E(1) = 4 \underbrace{\iint 1\cdot f(x,y)\,dy\,dx}_{=1} = 4\cdot 1 = 4$$

Merk:

$$E(U\cdot V) \neq E(U)\cdot E(V)$$

Regneregler for varians:

$$\text{Var}(aX+bY) = E\left((aX+bY)^2\right) - E(aX+bY)^2$$

$$= E\left[a^2X^2 + 2abXY + b^2Y^2\right] - \left(aE(X)+bE(Y)\right)^2$$

$$= a^2E(X^2) + 2ab\,E(XY) + b^2E(Y^2) - \left(a^2E(X)^2 + 2ab\,E(X)E(Y) + b^2E(Y)^2\right)$$

$$= \underbrace{a^2E(X^2) - a^2E(X)^2}_{a^2\cdot\text{Var}(X)} + \underbrace{2ab\,E(XY) - 2ab\,E(X)E(Y)}_{2ab\cdot\text{Cov}(X,Y)} + b^2\cdot\text{Var}(Y)$$

Konkl: $\text{Var}(aX+bY) = a^2\,\text{Var}(X) + 2ab\,\text{Cov}(X,Y) + b^2\,\text{Var}(Y)$

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\,\text{Cov}(X,Y)$$

Regneregler for Covarians:

Eks: $\text{Cov}(X+Y, X-2Y)$

$$= \text{Cov}(X,X) + \text{Cov}(X,-2Y) + \text{Cov}(Y,X) + \text{Cov}(Y,-2Y)$$

$$= \text{Var}(X) - 2\,\text{Cov}(X,Y) + \text{Cov}(X,Y) - 2\,\text{Var}(Y)$$

$$= \text{Var}(X) - \text{Cov}(X,Y) - 2\,\text{Var}(Y)$$

i) Cov er bilineær

ii) $\text{Cov}(U,U) = \text{Var}(U)$

iii) $\text{Cov}(X,Y) = \text{Cov}(Y,X)$

Eks: Vi velger en tilfeldig valgt mann

X = avkastning til Aksje 1

Y = — " — Aksje 2

Vi velger en portefølje med vektene w_1, w_2.

$R = w_1X + w_2Y$ porteføljeavkastning

$$E(R) = E(w_1X + w_2Y) = w_1 \cdot E(X) + w_2 \cdot E(Y)$$

$$Var(R) = Var(w_1X + w_2Y) = w_1^2 Var(X) + 2w_1w_2 Cov(X,Y) + w_2^2 Var(Y)$$

$\underline{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$: $E(R) = (E(X)\ E(Y)) \cdot \underline{w}$

$Var(R) = \underline{w}^T \Sigma \underline{w}$, der

$$\Sigma = \begin{pmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{pmatrix}$$

er kovariansmatrisen

kvadr. fukjon i $\underline{x}$:

$\underline{x}^T A \underline{x} + B\underline{x} + C$

Merk: Kovariansmatrisen Σ er alltid positive semidefinit.

$\Uparrow$

$Var(R) = \underline{w}^T \Sigma \underline{w} \geq 0$ for alle $\underline{w}$

$\parallel$

$E[(R - \mu_R)^2]$

(3) Uavhengighet

Defn: X og Y er uavhengige stok. variabler hvis $f(x,y) = f_X(x) \cdot f_Y(y)$

Anta X, Y er uavhengige: $\mathrm{Cov}(X,Y) = ?$

$$E(XY) = \int\int xy \cdot f(x,y)\, dy\, dx = \int\int xy \cdot f_X(x) \cdot f_Y(y)\, dy\, dx$$

$$= \int x f_X(x) \underbrace{\int y f_Y(y)\, dy}_{= E(Y)}\, dx = \int x \cdot f_X(x) \cdot E(Y)\, dx$$

$$= E(Y) \cdot \underbrace{\int x f_X(x)\, dx}_{E(X)} = E(X) \cdot E(Y) \Rightarrow \mathrm{Cov}(X,Y) = 0$$

$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$

Resultat: $\boxed{X, Y \text{ uavhengige} \Rightarrow \mathrm{Cov}(X,Y) = 0}$

Merk: Selv om $\mathrm{Cov}(X,Y) = 0$, trenger ikke X og Y være uavhengige

Eks 1: $f(x,y) = x+y$, $0 \le x, y \le 1$

$f_X(x) = x + 1/2$, $0 \le x \le 1$

$f_Y(y) = y + 1/2$, $0 \le y \le 1$

$f_X(x) \cdot f_Y(y) = (x+1/2)(y+1/2)$

$= xy + \frac{1}{2}x + \frac{1}{2}y + \frac{1}{4} \neq f(x,y) = x+y$

$\mathrm{Cov}(X,Y) = -\frac{1}{144} \neq 0$

X, Y ikke uavhengige

Eks 2: $f(x,y) = 4xy$, $0 \le x,y \le 1$

$$f_x(x) = \int_0^1 4xy\,dy = \left[4x \cdot \tfrac{1}{2}y^2\right]_0^1 = \left[2xy^2\right]_0^1 = 2x, \quad 0 \le x \le 1$$

$$f_y(y) = \int_0^1 4xy\,dx = \left[4y \cdot \tfrac{1}{2}x^2\right]_0^1 = 2y, \quad 0 \le y \le 1$$

$$f_x(x) \cdot f_y(y) = 2x \cdot 2y = 4xy = f(x,y)$$

X og Y er uavhengige

Sjekk:

$$\int_0^1 \underbrace{\int_0^1 4xy\,dy}_{f_x(x)=2x} dx = \int_0^1 2x\,dx = \left[x^2\right]_0^1 = 1$$ (ok)