

Plan

- 1 Systems of differential equations
- 2 Introduction to difference equations
- 3 Systems of difference equations

Review: ~ equilibrium states and their stability
for first order autonomous differential
equation: $y' = F(y)$

~ second order linear differential equation
 $y'' + ay' + by = f(t)$

Superposition principle: $y = y_h + y_p$

① Systems of differential equations

Ex:
$$\begin{aligned} y_1' &= 2y_1 + 5y_2 \\ y_2' &= 5y_1 + 2y_2 \end{aligned}$$

coupled
system

Ex:
$$\begin{aligned} y_1' &= 3y_1 - y_2 \\ y_2' &= \end{aligned}$$

decoupled
system

$$A = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

$$y_1' = 3y_1$$

$$y_1' - 3y_1 = 0$$

$$\begin{aligned} y_1 &= y_1^h + y_1^p \\ &= C_1 \cdot e^{3t} + 0 \end{aligned}$$

$$\left. \begin{aligned} & \\ & \end{aligned} \right\} y_1 = \underline{C_1 e^{3t}}$$

$$\begin{aligned} y_2' &= -y_2 & y_2' + y_2 &= 0 \\ y_2 &= C_2 \cdot e^{-t} \end{aligned}$$

$$\left. \begin{aligned} & \\ & \end{aligned} \right\} y_2 = \underline{C_2 e^{-t}}$$

Char. eqn: $r - 3 = 0$
 $r = 3$
 $\hat{=} e^{3t}$ solution

Defn: A system of first order differential equations in $y_1, y_2, \dots, y_n$ has the form

$$y_1' = F_1(t, y_1, y_2, \dots, y_n)$$

$$y_2' = F_2(t, y_1, y_2, \dots, y_n)$$

$$\vdots$$

$$y_n' = F_n(t, y_1, y_2, \dots, y_n)$$

It is called autonomous if $F_1, \dots, F_n$ are independent of t .

A linear system of first order differential equations has the form

$$y_1' = a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n + b_1$$

$$y_2' = a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n + b_2$$

$$\vdots$$

$$y_n' = a_{n1}y_1 + a_{n2}y_2 + \dots + a_{nn}y_n + b_n$$

where a_{ij}, b_i are constants. It is called homogeneous if $b_1 = b_2 = \dots = b_n = 0$.

Ex:

$$y_1' = 2y_1 + 5y_2 - 1$$

$$y_2' = 5y_1 + 2y_2 + 3$$

Remark:

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \text{ state vector}$$

$$\underline{y}' = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix}$$

Matrix notation:

$$\underline{y}' = A \cdot \underline{y} + \underline{b}$$

where A is an $n \times n$ -matrix, $\underline{b}$ is an n -vector.

Homogeneous case:

$$\underline{y}' = A \cdot \underline{y}$$

|| decouple the system

Assume that $\underline{b} = \underline{0}$ and that A is diagonalizable:
There are n linearly independent eigenvectors $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ of A with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ s.t.
 $A \cdot \underline{v}_i = \lambda_i \underline{v}_i$ for $1 \leq i \leq n$.

New variables: $\underline{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$

$$A \cdot \underline{v}_i = \lambda_i \underline{v}_i \text{ for } 1 \leq i \leq n.$$

$$\underline{P}^{-1} A \underline{P} = D$$

when $\underline{P} = (\underline{v}_1 | \dots | \underline{v}_n)$

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

$$\underline{y} = \underline{P} \cdot \underline{z} \Leftrightarrow \underline{z} = \underline{P}^{-1} \cdot \underline{y}$$

$$\underline{z}' = (\underline{P}^{-1} \cdot \underline{y})' = \underline{P}^{-1} \cdot \underline{y}'$$

$$= \underline{P}^{-1} \cdot (A \underline{y}) = \underline{P}^{-1} A \underline{y}$$

$$= \underline{P}^{-1} A \cdot (\underline{P} \underline{z}) = \underline{P}^{-1} A \underline{P} \cdot \underline{z} = D \underline{z}$$

$$\underline{z}' = D \cdot \underline{z}$$

decoupled system

$$\begin{pmatrix} z_1' \\ z_2' \\ \vdots \\ z_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

$$\begin{aligned} z_1' &= \lambda_1 \cdot z_1 \\ z_2' &= \lambda_2 \cdot z_2 \\ &\vdots \\ z_n' &= \lambda_n \cdot z_n \end{aligned}$$

General solution:

$$z_1 = C_1 \cdot e^{\lambda_1 t}$$

$$z_2 = C_2 \cdot e^{\lambda_2 t}$$

$\vdots$

$$z_n = C_n \cdot e^{\lambda_n t}$$

$$\begin{aligned} z_1' - \lambda_1 z_1 &= 0 && \text{lin. homog.} \\ z_1 &= C_1 \cdot e^{r_1 t} && r_1 = \lambda_1 \\ &= C_1 \cdot e^{\lambda_1 t} \end{aligned}$$

$$\Rightarrow \underline{z} = \begin{pmatrix} C_1 e^{\lambda_1 t} \\ C_2 e^{\lambda_2 t} \\ \vdots \\ C_n e^{\lambda_n t} \end{pmatrix}$$

$$\underline{y} = \underline{P} \underline{z} = (\underline{v}_1 | \underline{v}_2 | \dots | \underline{v}_n) \cdot \begin{pmatrix} C_1 e^{\lambda_1 t} \\ C_2 e^{\lambda_2 t} \\ \vdots \\ C_n e^{\lambda_n t} \end{pmatrix}$$

$$\underline{y} = C_1 \underline{v}_1 e^{\lambda_1 t} + C_2 \underline{v}_2 e^{\lambda_2 t} + \dots + C_n \underline{v}_n e^{\lambda_n t}$$

Summary: Homogeneous case

$$\left. \begin{array}{l} \underline{y}' = A \underline{y} \\ \text{with } A \\ \text{diagonalizable} \end{array} \right\} \Rightarrow$$

General solution:

$$\underline{y} = C_1 \underline{u}_1 e^{\lambda_1 t} + C_2 \underline{u}_2 e^{\lambda_2 t} + \dots + C_n \underline{u}_n e^{\lambda_n t}$$

Ex:

$$\begin{aligned} y_1' &= 2y_1 - 5y_2 \\ y_2' &= 5y_1 + 2y_2 \end{aligned}$$

homogeneous in
system of diff.-eqns.

$$A = \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix} \quad \begin{array}{l} \text{symmetric} \\ \text{diagonalizable} \end{array}$$

$$\underline{y}' = A \underline{y}$$

i) Eigenvalues:

$$\begin{vmatrix} 2-\lambda & 5 \\ 5 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 21 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16 + 84}}{2} = 2 \pm 5$$

$$\lambda_1 = 2, \quad \lambda_2 = -3$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$

ii) Eigenvectors:

$$\lambda = 2: \begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix} \rightarrow \underline{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = -3: \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \rightarrow \underline{u}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

General solution:

$$\underline{y} = C_1 \underline{u}_1 e^{\lambda_1 t} + C_2 \underline{u}_2 e^{\lambda_2 t}$$

$$= C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$$

$$\begin{aligned} y_1(t) &= C_1 e^{2t} - C_2 e^{-3t} \\ y_2(t) &= C_1 e^{2t} + C_2 e^{-3t} \end{aligned}$$

Connection with second order linear differential equations: Homogeneous case

$$\boxed{y'' + ay' + by = 0} \Rightarrow y'' = -by - ay'$$

use let: $y_1 = y$
 $y_2 = y'$

$$\begin{aligned} y_1' &= y' = y_2 \\ y_2' &= y'' = -by - ay' = -by_1 - ay_2 \end{aligned}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \underline{A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}}$$

i) Second order diff-equ:

$$y = y_h + y_p = y_h$$

Char.
eqn.

$$\boxed{r^2 + ar + b = 0}$$

ii) Linear system:

$$y' = Ay \quad A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -b & -a-\lambda \end{vmatrix} = 0$$

$$-\lambda(-a-\lambda) + b = 0$$

Same eqn. ! $\Rightarrow \underline{\lambda^2 + a\lambda + b = 0}$

② Difference equations

Defn: A first order difference equation is an equation which involves y_{t+1} and y_t , and the unknown is a sequence (y_t) .

Ex: $y_{t+1} = 1.05 y_t$ or $y_{t+1} - 1.1 y_t = 10$

Sequences: $(y_t) : y_0, y_1, y_2, y_3, \dots$
a numerical value for all non-negative integers $t = 0, 1, 2, \dots$

y_t : sequence
 $y(t)$: function
(cont.)

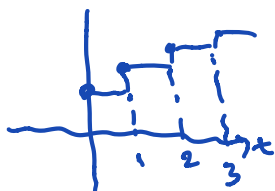
Ex: $y_0 = 1, y_1 = 2, y_2 = 4, y_3 = 8, \dots$
 $y_t = 2^t, t \geq 0$ integer

Ex: $y_{t+1} = 1.05 y_t$
 $y_{t+1} - y_t = 0.05 y_t$
change in the sequence

General form:

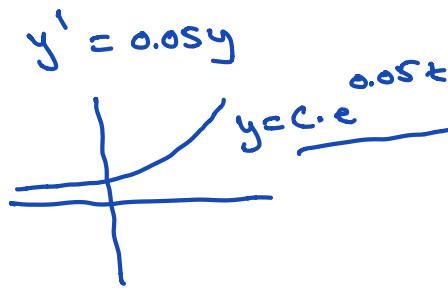
$y_{t+1} = F(t, y_t)$
 $\Leftrightarrow$
 $y_{t+1} - y_t = F(t, y_t) - y_t$
change

Compare: $y_{t+1} = 1.05 y_t$
 $y_{t+1} - y_t = 0.05 y_t$



$y_1 = 1.05 y_0 \quad (t=0)$
 $y_2 = 1.05 y_1 \quad (t=1)$
 $= 1.05 (1.05 y_0) = 1.05^2 y_0$

$y = C \cdot 1.05^t, t = 0, 1, 2, \dots$
 $C = y_0$



The general solution of a first order difference equation will depend on one undetermined coeff.

i) Linear first order difference equations:

$$y_{t+1} + a \cdot y_t = b_t$$

a: constant
 b_t : sequence

← Superposition principle:

$$y_t = y_t^h + y_t^p$$

y_t^h : general solution of the homogeneous eqn.
 $y_{t+1} + a y_t = 0$

y_t^p : particular solution of $y_{t+1} + a y_t = b_t$

Ex1 $y_{t+1} - 1.10 y_t = 10$

$$y_t = y_t^h + y_t^p = \underline{C \cdot 1.10^t - 100}$$

y_t^h : $y_{t+1} - 1.10 y_t = 0$

$$r - 1.10 \cdot 1 = 0$$

$$r = 1.10$$

$$\Rightarrow y_t^h = \underline{C \cdot 1.10^t}$$

$$(y_t = C \cdot r^t)$$

use $C \cdot r^t$
 (not $C \cdot e^{rt}$)

Alt: $y_{t+1} = 1.10 \cdot y_t$

$$y_1 = 1.10 \cdot y_0$$

$$y_2 = 1.10 y_1 = 1.10^2 y_0$$

$$y_3 = 1.10 \cdot 1.10^2 y_0 = 1.10^3 y_0$$

$\vdots$

$$y_t = \underline{C \cdot 1.10^t}$$

y_t^p : $y_{t+1} - 1.10 y_t = 10$

$$b_t = 10$$

$$b_{t+1} = 10$$

$$\underline{y_t = A}$$

$$A - 1.10 A = 10$$

$$-0.10 A = 10$$

$$A = \frac{10}{-0.10} = -100$$

$$\underline{y_t^p = -100}$$

ii) Linear second order difference equations.

$$y_{t+2} + a y_{t+1} + b y_t = f_t$$

a, b : constants
 f_t : expr. in t

Superposition principle:

$$y_t = y_t^h + y_t^p$$

Ex: $y_{t+2} - 7y_{t+1} + 12y_t = t$

$$y_t = y_t^h + y_t^p = \underbrace{C_1 \cdot 3^t + C_2 \cdot 4^t}_{y_t^h} + \underbrace{\frac{1}{6}t + \frac{5}{36}}_{y_t^p}$$

y_t^h : $y_{t+2} - 7y_{t+1} + 12y_t = 0$

$$r^2 - 7r + 12 = 0$$

$$r_1 = 3, r_2 = 4$$

$$\Rightarrow y_t^h = C_1 \cdot r_1^t + C_2 \cdot r_2^t = \underline{C_1 \cdot 3^t + C_2 \cdot 4^t}$$

y_t^p : $y_{t+2} - 7y_{t+1} + 12y_t = \underline{t}$

$$\left. \begin{aligned} b_t &= t \\ b_{t+1} &= t+1 \\ b_{t+2} &= t+2 \end{aligned} \right\}$$

$$\begin{aligned} y_t &= At + B \\ y_{t+1} &= A(t+1) + B \\ &= At + (A+B) \\ y_{t+2} &= A(t+2) + B \\ &= At + (2A+B) \end{aligned}$$

If $r_1 = r_2$:

$$y_t^h = C_1 \cdot r_1^t + C_2 \cdot t \cdot r_1^t$$

If there are no real roots:

solutions in terms of $\sin / \cos$

$$\begin{aligned} At + (2A+B) - 7(At + A+B) + 12(At + B) &= t \end{aligned}$$

$$\underline{6At} + \underline{(-5A+6B)} = \underline{t}$$

$$\underline{y_t^p = \frac{1}{6}t + \frac{5}{36}}$$

$$6A = 1 \quad A = \underline{1/6}$$

$$-5A + 6B = 0$$

$$\underline{\frac{6B}{6} = 5A = 5/6}$$

$$B = \underline{5/36}$$

Note: $y_{t+1} - y_t$: change

$\Delta y_{t+1} - \Delta y_t$: second order change

$$\Delta y_t = y_{t+1} - y_t$$

$$\begin{aligned} \Delta y_{t+1} - \Delta y_t &= (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t) \\ &= y_{t+2} - 2y_{t+1} + y_t \end{aligned}$$

$$\underline{y_{t+2} = F(t, y_t, y_{t+1})}$$

$$y_{t+2} - 2y_{t+1} + y_t = F(t, y_t, y_{t+1}) - 2y_{t+1} + y_t$$

③ Systems of difference equations:

Ex:

$$\begin{aligned} y_{1,t+1} &= 0.85 y_{1,t} + 0.14 y_{2,t} \\ y_{2,t+1} &= 0.15 y_{1,t} + 0.86 y_{2,t} \end{aligned}$$

$$\underline{y}_{t+1} = \begin{pmatrix} 0.85 & 0.14 \\ 0.15 & 0.86 \end{pmatrix} \underline{y}_t = A \cdot \underline{y}_t$$

2x2 system of linear
difference equation

In general: If $\underline{y}_{t+1} = A \cdot \underline{y}_t$ and A is a diagonalizable
 $n \times n$ -matrix with linearly independent eigenvector
 $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ of A with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$

$$P = (\underline{v}_1 | \underline{v}_2 | \dots | \underline{v}_n) \quad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

then:

$$\underline{y}_t = C_1 \underline{v}_1 \cdot \lambda_1^t + C_2 \underline{v}_2 \cdot \lambda_2^t + \dots + C_n \underline{v}_n \lambda_n^t$$

is the general solution of the system.

Ex: $\underline{y}_{t+1} = A \cdot \underline{y}_t$ with $A = \begin{pmatrix} 0.85 & 0.14 \\ 0.15 & 0.86 \end{pmatrix}$

$$\begin{vmatrix} 0.85 - \lambda & 0.14 \\ 0.15 & 0.86 - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - 1.71\lambda + 0.71 = 0$$

$$\underline{\lambda_1 = 1}, \quad \underline{\lambda_2 = 0.71}$$

E1: $\begin{pmatrix} -0.15 & 0.14 \\ 0.15 & -0.14 \end{pmatrix}$

$$-0.15x + 0.14y = 0$$

$$-15x + 14y = 0$$

$$x = \frac{14}{15}y, \quad y \text{ free}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{14}{15}y \\ y \end{pmatrix} = \frac{y}{15} \begin{pmatrix} 14 \\ 15 \end{pmatrix}$$

$$\underline{v_1 = \begin{pmatrix} 14 \\ 15 \end{pmatrix}}$$

E021: $\begin{pmatrix} 0.14 & 0.14 \\ 0.15 & 0.15 \end{pmatrix} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix}$

$$14x + 14y = 0$$

$$x = -y, \quad y \text{ free}$$

$$= y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \underline{v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

General solution, $y_t = c_1 \cdot \begin{pmatrix} 14 \\ 15 \end{pmatrix} \cdot 1^t + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot 0.71^t$

$$= \underline{\underline{c_1 \cdot \begin{pmatrix} 14 \\ 15 \end{pmatrix} + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot 0.71^t}}$$