Plan

1 Unconstrained optimization
2 Convex and concave functions

Review:

- Markov chains: A regular Markov chain matrix $\Rightarrow$ $\lambda = 1$ eigenvalue of A (mult. 1), there is unique eigenvector $\underline{v}$ in E_1 which is a state vector
$\Updownarrow$
$\lim_{m \to \infty} A^m \underline{v}_0 = \underline{v}$ equilibr. state

$A \cdot \underline{v}_i = \underline{v}_{i+1}$

- quadratic form

$f(\underline{x}) = \underline{x}^T A \underline{x}$ (A symmetric matrix)

D_i: leading principal minors

- definiteness:

pos. defn $\Leftrightarrow$ $D_1, D_2, \ldots, D_n > 0$
neg. defn. $\Leftrightarrow$ $D_1 < 0, D_2 > 0, \ldots$

all other cases: indefinite $\rightarrow$

pos. semidefn. $\Leftrightarrow$ $\Delta_1, \Delta_2, \ldots, \Delta_n \geq 0$
neg. — " — $\Leftrightarrow$ $\Delta_1 \leq 0, \Delta_2 \geq 0, \ldots$

Δ_i: any principal minor

- reduced rank criterion

① $\text{rk } A = r < n$
② $D_1, D_2, \ldots, D_r > 0$

RRC $\Rightarrow$ A pos. semidefn.

① $\text{rk } A = r < n$
② $D_1 < 0, D_2 > 0, \ldots, (-1)^r D_r > 0$

RRC $\Rightarrow$ A neg. semi-defn.

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1 Unconstrained optimization

Forkloop: Lecture 6, Part 1–3

$$\max/\min \ f(x_1, x_2, \ldots, x_n) = f(\underline{x})$$

Method:

i) Find all stationary pts of f

ii) Classify them as local max, local min or saddle pts

iii) Use this information + additional information to conclude about global max/min = max/min

Defn: $\underline{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$

$\underline{x} = \underline{x}^*$ is a (global) max for f if $f(\underline{x}^*) \geq f(\underline{x})$ for all pts $\underline{x}$

$\underline{x} = \underline{x}^*$ is a (global) min for f if $f(\underline{x}^*) \leq f(\underline{x})$ for all pts $\underline{x}$

$\underline{x} = \underline{x}^*$ is local max for f if $f(\underline{x}^*) \geq f(\underline{x})$ for all $\underline{x}$ close to $\underline{x}^*$

$\underline{x} = \underline{x}^*$ is local min for f if $f(\underline{x}^*) \leq f(\underline{x})$ for all $\underline{x}$ close to $\underline{x}^*$

i) Stationary pts:

Defn: A stationary pt for f is a pt where

$$\boxed{f'_{x_1} = f'_{x_2} = \dots = f'_{x_n} = 0}$$

FOC = first order equ's.

Ex: $f(x,y) = x^2y^3 + y^2 - 2y$

FOC: $\begin{cases} f'_x = 2xy^3 = 0 \\ f'_y = x^2 \cdot 3y^2 + 2y - 2 = 0 \end{cases}$

$x=0$ or	$y=0$
$2y-2=0$	$-2=0$
$y=1$	not any sol's
$\Downarrow$	
$(0,1)$	

Stationary pts: $(x,y) = (0,1)$

Fact: If f is "nice", then any max/min for f is a stationary pt.

Candidate pts = stationary pts

ii) Local classification.

Defn: A saddle pt is a stationary pt that is not a local max or a local min.

Second derivative test:

If $\underline{x}^*$ is a stationary pt, consider the Hessian

$$H(f)(\underline{x}^*) = \begin{pmatrix} f''_{x_1x_1}(\underline{x}^*) & f''_{x_1x_2}(\underline{x}^*) & \dots & f''_{x_1x_n}(\underline{x}^*) \\ \vdots & \vdots & & \vdots \\ f''_{x_nx_1}(\underline{x}^*) & f''_{x_nx_2}(\underline{x}^*) & \dots & f''_{x_nx_n}(\underline{x}^*) \end{pmatrix}$$

$n \times n$ symmetric matrix

Then we have:

$H(f)(\underline{x}^*)$ positive definite $\Rightarrow$ $\underline{x}^*$ local min
$H(f)(\underline{x}^*)$ negative —"— $\Rightarrow$ $\underline{x}^*$ local max
$H(f)(\underline{x}^*)$ indefinite $\Rightarrow$ $\underline{x}^*$ saddle pt.

Otherwise, the test is inconclusive.

Ex: $f(x,y) = x^2y^3 + y^2 - 2y$

$f'_x = 2xy^3 = 0$
$f'_y = 3x^2y^2 + 2y - 2 = 0$

Stationary pts: $(x,y) = (0,1)$

Second derivative test:

$$H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} = \begin{pmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y+2 \end{pmatrix}$$

$$H(f)(0,1) = \begin{pmatrix} \overset{A}{2} & \overset{B}{0} \\ \underset{B}{0} & \underset{C}{2} \end{pmatrix}$$

$D_1 = 2 = A$
$D_2 = 4 = AC - B^2$

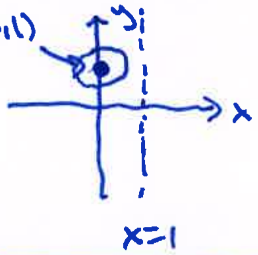
$H(f)(0,1)$ is pos. defin. $\xRightarrow{\text{second der. test}}$ $(x,y) = (0,1)$ is local min for f

ii) Is $(x,y) = (0,1)$ global min for f?

global min $\Rightarrow$ local min
global max $\Rightarrow$ local max

$f(0,1) = -1$ local min.

$-1 = f(0,1)$

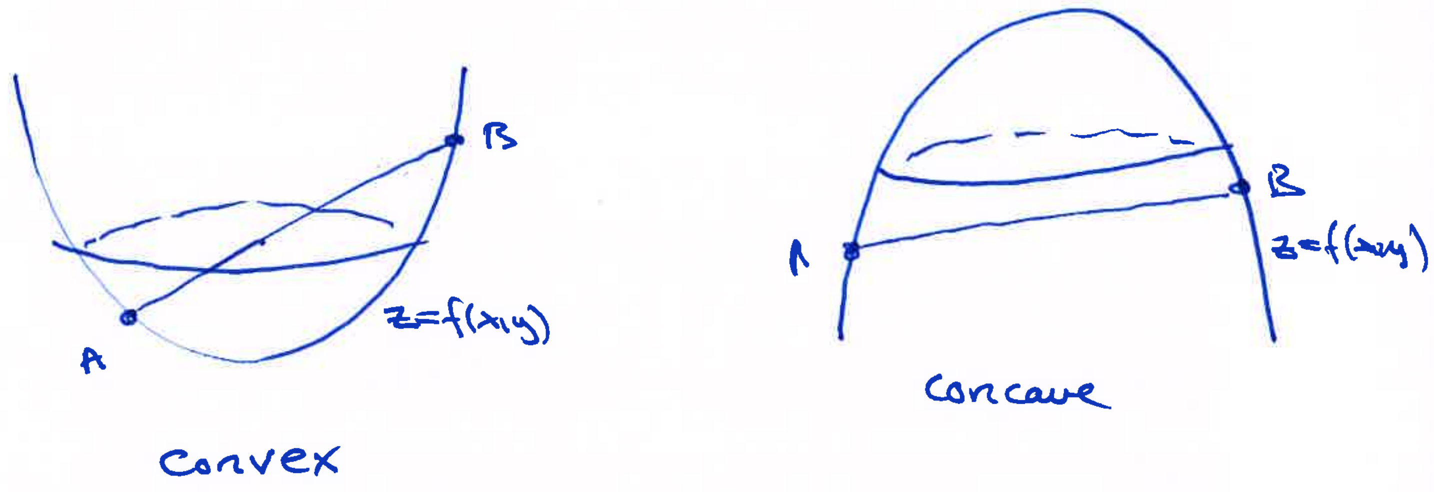


$f = x^2y^3 + y^2 - 2y$
$x=0$: $f(0,y) = y^2 - 2y$
$x=1$: $f(1,y) = y^3 + y^2 - 2y$

Not global min: $(0,1)$

$\boxed{\text{No global min or max}}$ $f(1,-2) = -8 + 4 + 4 = 0$ $\quad f(1,-3) = -27 + 9 + 6 = -12$

2 Convex and concave function



<u>Defn:</u> $f(x_1, \ldots, x_n)$ fn. in n variables

f is <u>convex</u> if: For any two pts A and B on the graph of f, then the line segment $[A, B]$ lies <u>over</u> the graph of f (or on the graph).

f is <u>concave</u> if: For any two pts A and B on the graph of f, the line segment $[A, B]$ lies <u>under</u> the graph of f (or on the graph)

<u>Ex:</u> $f(x,y) = x^2y^3 + xy^2 - 2y$

<u>Result:</u> Assume f is a "nice" function

f is convex $\iff$ $H(f)(\underline{x})$ is positive semidefinite for all $\underline{x}$
f is concave $\iff$ $H(f)(\underline{x})$ is negative semidefinite for all $\underline{x}$

$\boxed{\text{global result}}$

Ex: $f = x^2y^3 + y^2 - 2y$ Is f convex / concave?

$$f'_x = 2xy^3$$
$$f'_y = 3x^2y^2 + 2y - 2$$

$$H(f) = \begin{pmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y+2 \end{pmatrix}$$

$$D_1 = 2y^3$$
$$D_2 = 12x^2y^4 + 4y^3 - 36x^2y^4 = 4y^3 - 24x^2y^4$$

f convex $\iff$ $H(f)$ pos. semidefin. for all (x,y)

$\Updownarrow$

$\left.\begin{matrix} 2y^3 \geq 0 \\ \text{for all } (x,y) \end{matrix}\right\} \rightarrow$ $D_1 \geq 0$, $\Delta_1 \geq 0$; $D_2 \geq 0$, $\Delta_2 \geq 0$ No.

f concave $\implies$ $D_1 \leq 0$ for all (x,y) No.

$2y^3 \leq 0$ for all (x,y)

Convex / concave optimization:

f convex $\implies$ any stationary pt. is global min

f concave $\implies$ any stationary pt is global max

convex

concave

Plan:

1. Reduced rank criterion : example
2. Unconstrained optimization

1 Reduced rank criterion

Ex: $f(x,y,z,w) = x^2 + y^2 + 6yz - 4yw + 9z^2 - 12zw + 4w^2$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 9 & -6 \\ 0 & -2 & -6 & 4 \end{pmatrix}$$

$D_1 = 1$
$D_2 = 1$
$D_3 = 1 \cdot 0 = 0$
$D_4 = 1 \cdot \begin{vmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ -2 & -6 & 4 \end{vmatrix}$
$= 1 \cdot 0 - 3 \cdot 0 - 2 \cdot 0$
$= 0$

pos. semidefin. ?

Δ_1: $1, 1, 9, 4 \geq 0$
Δ_2: $1, 0, 0, 9, 0, 4 \geq 0$
12 23 34 13 24 34
Δ_3: $0, 0, 0, 0 \geq 0$
123 124 134 234
Δ_4: $0 \geq 0$

Alternative: RRC

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 9 & -6 \\ 0 & -2 & -6 & 4 \end{pmatrix} \begin{matrix} \\ \\ ]{-3} \\ ]{2} \end{matrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{rk}\, A = 2$

$D_1 = 1,\ D_2 = 1$

A pos. semidefinite

Ex: $f = 2xy - x^2 - 2y^2 + 2yz - z^2$

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$D_1 = -1 < 0$
$D_2 = 1 > 0$
$D_3 = -1(-1) + (-1) \cdot 1$
$= 1 - 1 = 0$

RRC: $\text{rk}\, A = 2$

$\left.\begin{matrix} D_1 = -1 \\ D_2 = 1 \end{matrix}\right\}$ A neg. semidefin.

② Unconstrained optimization:

Ex: $f(x,y,z) = x^4 + 2x^2 + y^2 + 2z^2 + z^4 - 4xz$

i) Stat. pts: $f'_x = 4x^3 + 4x - 4z = 0$

$f'_y = 2y = 0 \quad \leftarrow \quad y = 0$

$f'_z = 4z + 4z^3 - 4x = 0$

$x^3 + x - z = 0 \qquad x - z = -x^3$

$z^3 + z - x = 0 \qquad x - z = z^3$

$-x^3 = z^3$

$\sqrt[3]{-x^3} = \sqrt[3]{z^3}$

$-x = z$

$x^3 + x + x = 0$

$x^3 + 2x = 0 \qquad x(x^2+2) = 0$

$x = 0$ or $\cancel{x^2+2=0}$

$z = 0$

Stat. pts: $(x,y,z) = (0,0,0) \qquad f = 0$

ii) Local classification:

$$H(f) = \begin{pmatrix} 12x^2+4 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & 12z^2+4 \end{pmatrix}$$

$$H(f)(0,0,0) = \begin{pmatrix} 4 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & 4 \end{pmatrix}$$

$D_1 = 4$

$D_2 = 8$

$D_3 = 2 \cdot 0 = 0$

REC:

$rk = 2$

$D_1 = 4, D_2 = 8 > 0$

pos. semidefinite

Second derivative test: inconclusive

What can we do in these cases:

- check if convex / concave
- use defn of local max/min

Is f convex?

$$H(f) = \begin{pmatrix} 12x^2+4 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & 12z^2+4 \end{pmatrix}$$

$D_1 = 12x^2+4 > 0$ for all x,y,z

$D_2 = 2 \cdot D_1 > 0$ — " —

$D_3 = 2 \cdot [(12x^2+4)(12z^2+4) - 16]$

$= 2(144x^2z^2 + 48x^2 + 48z^2) \geq 0$

$D_3 > 0$: $D_1, D_2, D_3 > 0 \Rightarrow H(f)$ pos. defn.

$D_3 = 0$: $(x=z=0)$ $D_1, D_2 > 0$, $D_3 = 0$, $\text{Rk } H(f) = 2$ $\Rightarrow$ (RRC) $H(f)$ pos. semidefn.

for all x,y,z

f convex

$\Downarrow$

$(0,0,0)$ (local) and global min

Ex: $f(x,y,z) = x^3 + y^3 + z^3 - 3xz$

$f'_x = 3x^2 - 3z = 0$ $\quad z = x^2$

$f'_y = 3y^2 = 0$ $\quad y = 0$

$f'_z = 3z^2 - 3x = 0$ $\quad x = z^2 = (x^2)^2 = x^4$

$x = x^4$

$x - x^4 = 0$

$x(1-x^3) = 0$

$x = 0$, $z = 0$, $y = 0$ or $x = 1$, $z = 1$, $y = 0$

$$\boxed{\text{Stat. pts: } (x,y,z) = (0,0,0),\ (1,0,1) \qquad f=0,\ f=-1}$$

$$H(f) = \begin{pmatrix} 6x & 0 & -3 \\ 0 & 6y & 0 \\ -3 & 0 & 6z \end{pmatrix}$$

$(0,0,0)$: $H(f)(0,0,0) = \begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{pmatrix}$

$D_1 = 0$, $D_2 = 0$, $D_3 < 0$

$\Delta_1: 0, 0, 0$

$\Delta_2 = 0, 0, -9$

indefinite

$\Downarrow$

$(0,0,0)$ saddle pt

$H(f)(1,0,1) = \begin{pmatrix} 6 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 6 \end{pmatrix}$

$D_1 = 6$, $D_2 = 0$, $D_3 = 0$

$\Delta_1: 6, 0, 6 \geq 0$

$\Delta_2: 0, 0, 27 \geq 0$

$\Delta_3: 0 \geq 0$

pos. semidefn.

inconclusive sec. der. test

f not convex/concave

Use defn. $\boxed{\text{local max/min/saddle pt.}}$

$f(1,0,1) = -1$

stationary pt.

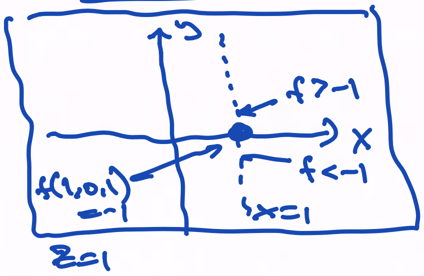
local min

$\Uparrow$

$f(x,y,z) \geq -1$ close to $(1,0,1)$

$f(x,y,z) = x^3+y^3+z^3-3xz$

Look at $x=z=1$:



(1,0,1) saddle pt.

$$f(1,y,1) = 1^3+y^3+1^3-3$$
$$= -1+y^3$$