Plan

1 Optimal control theory
2 Applications: Portfolio optimization

① Optimal control theory

not relevant for the final exam

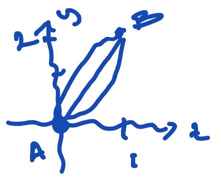
A functional is a rule that to a function $y = y(t)$ associates a number: $y \longrightarrow J(y)$

Ex: $J(y) = \int_0^1 y(t)\,dt$

$J(y) = \int_0^3 y(t)^2 + \dot{y}(t)^2\,dt$

Variational calculus:

max/min $J(y) = \int_a^b F(t, y, y')\,dt$ when $\begin{cases} y(a) = y_0 \\ y(b) = y_1 \end{cases}$



$a = 0 \quad y_0 = 0$
$b = 1 \quad y_1 = 2$

Ex: min $\int_0^1 y(t)^2 + y'(t)^2\,dt$ when $\begin{cases} y(0) = 0 \\ y(1) = 2 \end{cases}$

Euler equation: Necessary condition for max/min

$$F'_y - \frac{d}{dt}\left(F'_{y'}\right) = 0$$

Ex: $F = y^2 + \dot{y}^2 = y^2 + u^2$ $\quad \dot{y} = y'$, $u = \dot{y}$

$F'_y = 2y$

$F'_{\dot{y}} = 2\dot{y}$ $\quad \frac{d}{dt}(F'_{\dot{y}}) = \frac{d}{dt}(2y') = 2y''$

Euler: $2y - 2y'' = 0 \quad | : (-2)$

$y'' - y = 0$ second order linear diff. eqn.

Char. eqn: $r^2 - 1 = 0$

$r = \pm 1 \Rightarrow y = C_1 e^t + C_2 e^{-t}$

Initial conditions:

$y(0) = 0: \quad C_1 + C_2 = 0$

$y(1) = 2: \quad C_1 \cdot e + C_2 \cdot \frac{1}{e} = 2$

$C_2 = -C_1$

$C_1 \cdot e - C_1 \cdot \frac{1}{e} = 2 \quad | \cdot e$

$C_1 e^2 - C_1 = 2e$

$C_1 (e^2 - 1) = 2e$

$$C_1 = \frac{2e}{e^2-1}, \quad C_2 = -\frac{2e}{e^2-1} \quad \Bigg\} \quad y(t) = \frac{2e}{e^2-1}e^t - \frac{2e}{e^2-1}e^{-t}$$

candidate for min.

Convexity / concavity:

$(y, y') \to F(y, y')$ convex $\Rightarrow$ any solution of the Euler eqn. is min

— " — concave $\Rightarrow$ any solution of the Euler eqn. is max

Ex: $F = y^2 + \dot{y}^2$

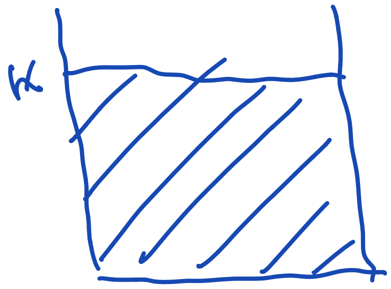
$$\left.\begin{array}{lll} F'_y = 2y & F''_{yy} = 2 & F''_{y\dot{y}} = 0 \\ F'_{\dot{y}} = 2\dot{y} & F''_{\dot{y}y} = 0 & F''_{\dot{y}\dot{y}} = 2 \end{array}\right\} \quad H(F) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

positive def. ($D_1 = 2$, $D_2 = 4$)

$$y^*(t) = \frac{2e}{e^2-1}\left(e^t - e^{-t}\right) \Leftarrow F \text{ convex in } (y, \dot{y})$$

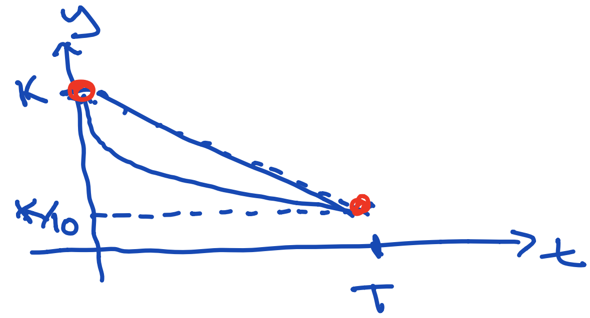
is a min

Ex:



oil reservoar

K units of oil



$y(t)$: no. units of oil left at time t

$u(t) = -y'(t)$: control variable

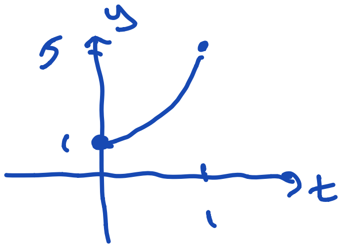
$$\max \int_0^T p(t)\cdot u(t) - C(t,y,u)\,dt$$

when $\begin{cases} y(0) = K \\ y(T) = K/10 \\ u = -y' \end{cases}$

Alt. I: use variational calculus (Euler's eqn.) : replace u with $-y'$

Alt. 2: use Pontryagin's maximum principle

Ex: $\min \int_0^1 \sqrt{1+u^2}\,dt$ when $\begin{cases} y(0)=1 \\ y(1)=5 \\ y'=u \end{cases}$



$$\max/\min \int_a^b F(t,y,u)\,dt \quad \text{when} \quad \begin{cases} y(a) = y_0 \\ y(b) = y_1 \\ y' = G(t,y,u) \end{cases}$$

Method: $H = p_0 \cdot F(t,y,u) + p \cdot G(t,y,u)$ (Hamiltonian)

$= p_0 \cdot \sqrt{1+u^2} + p \cdot u$

Necessary conditions: (for y^*, u^*)

$$\boxed{\begin{array}{l} \text{i) } p_0 = 0 \text{ or } p_0 = 1, \text{ and } (p_0,p) \neq (0,0) \\ \text{ii) } u = u^* \text{ defines a max for } H(y^*,u) \quad \boxed{H'_u = 0} \\ \text{iii) } p'(t) = -H'_y \end{array}}$$

Ex: $H = p_0 \cdot \sqrt{1+u^2} + pu$

i) Case $p_0 = 1$: $H = \sqrt{1+u^2} + pu$, $u \in \mathbb{R}$

$$H'_u = 0: \frac{1}{2\sqrt{1+u^2}} \cdot 2u + p = 0$$

$$\frac{\cancel{2}u}{\cancel{2}\sqrt{1+u^2}} = -p \Rightarrow p = -\frac{u}{\sqrt{1+u^2}}$$

$$p' = -H'_y = 0 \Rightarrow p(t) = c$$

$$\parallel$$

$$c = -\frac{u}{\sqrt{1+u^2}}$$

$$c \cdot \sqrt{1+u^2} = -u$$

$$c^2 \cdot (1+u^2) = u^2$$

$$c^2 + c^2u^2 = u^2$$

$$u^2 - c^2u^2 = c^2$$

$$\frac{u^2(1-c^2)}{1-c^2} = \frac{c^2}{1-c^2}$$

$$u^2 = \frac{c^2}{1-c^2}$$

$$u = \frac{\pm c}{\sqrt{1-c^2}} = \frac{-c}{\sqrt{1-c^2}}$$

$$u(t) = \frac{-c}{\sqrt{1-c^2}}$$

$$y' = u = \frac{-c}{\sqrt{1-c^2}}$$

$$y(t) = -\frac{c}{\sqrt{1-c^2}}t$$

ii) Case $p_0 = 0$: $H = pu$

$H'_u = 0$: $p = 0$

no candidate functions in this case

More material on optimal control theory: [FMEA] Further math. for economic analysis. Sydsæter et al.

(2) Portfolio optimization

Construct portfolio:

1 2 3 4 ... n assets

R_1 R_2 R_3 R_4 — R_n returns of asset i (stochastic variables)

Assumptions:

$E(R_i) = \mu_i$ known constant

$Var(R_i) = \sigma_i^2$ — " —

$Cov(R_i, R_j) = \sigma_{ij}$ — " —

Choose portfolio:

$w_1, w_2, \ldots, w_n$ portfolio weights

$w_1 + w_2 + \ldots + w_n = 1$

(allow $w_i < 0$: short selling)

Ex: A and B (shares) equal weight

$w_1 = 1/2$ $w_2 = 1/2$

Formulas: $R = w_1 R_1 + w_2 R_2 + \ldots + w_n R_n$ return of the portfolio

$$\underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}$$

$$E(R) = w_1 E(R_1) + w_2 E(R_2) + \ldots + w_n E(R_n)$$
$$= w_1\mu_1 + w_2\mu_2 + \ldots + w_n\mu_n$$
$$= (\mu_1\ \mu_2 \ldots \mu_n) \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \underline{\mu}^T \cdot \underline{w}$$

$$\Sigma = \begin{pmatrix} Var(R_1) & Cov(R_1,R_2) & \cdots \\ Cov(R_2,R_1) & Var(R_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

covariance matrix

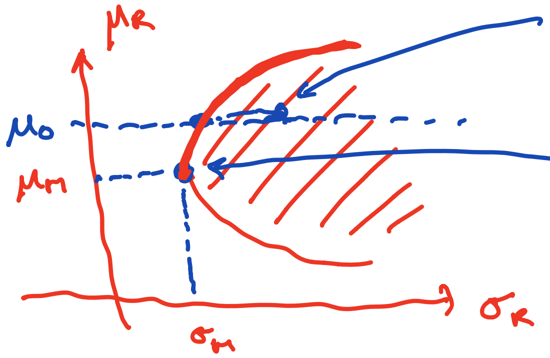
$$= \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{2n} & \cdots & \sigma_n^2 \end{pmatrix}$$

$$Var(R) = Cov(w_1R_1 + \ldots + w_nR_n,\ w_1R_1 + \ldots + w_nR_n)$$
$$= w_1^2\, Var(R_1) + w_1w_2\, Cov(R_1,R_2) + \ldots$$
$$\ldots + w_n^2\, Var(R_n)$$
$$= (w_1 \ldots w_n) \begin{pmatrix} Var(R_1) & \cdot & \\ & Var(R_2) & \\ \cdot & & \ddots \\ & & & Var(R_n) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \underline{w}^T \cdot \Sigma \cdot \underline{w}$$

$$\Downarrow$$

$$E(R) = \underline{\mu}^T \cdot \underline{w} = \mu_R$$
$$Var(R) = \underline{w}^T \Sigma \underline{w} = \sigma_R^2$$

we want μ_R max. and σ_R min.



pt (σ_R, μ_R) for a possible portfolio

a) minimum variance portfolio

$$\min \sigma_R^2 = \underline{w}^T \Sigma \underline{w} \quad \text{when} \quad \underline{u}^T \cdot \underline{w} = 1$$

b) minimum variance with given expect return μ_0.

$$\min \sigma_R^2 = \underline{w}^T \Sigma \underline{w} \quad \text{when} \quad \begin{cases} \underline{u}^T \cdot \underline{w} = 1 \\ \underline{\mu}^T \cdot \underline{w} = \mu_0 \end{cases}$$

$$\underline{u} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} : \quad \underline{u}^T \cdot \underline{w} = 1$$

$$(1\ 1\ \dots\ 1) \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = 1$$

Additional assumptions:

i) Σ is a positive definite (symmetric) matrix $\quad |\Sigma| \neq 0$

ii) $\{\underline{\mu}, \underline{u}\}$ are lin. independent

i) $Var(R) = \underline{w}^T \Sigma \underline{w} \geq 0$

$\Leftrightarrow \Sigma$ positive semidefinite (symm.)

pos. defn. $\Rightarrow |\Sigma| \neq 0$.

ii) $\{\underline{\mu}, \underline{u}\}$ lin. indep. $\Leftrightarrow \underline{\mu} \neq c \cdot \underline{u} = \begin{pmatrix} c \\ c \\ \vdots \\ c \end{pmatrix}$

Solution of a): $\quad \min \underline{w}^T \Sigma \underline{w} \quad$ when $\quad \underline{u}^T \cdot \underline{w} = 1$

Lagrange problem: $\quad \mathcal{L} = \underline{w}^T \Sigma \underline{w} - \lambda (\underline{u}^T \cdot \underline{w})$

FOC: $\quad \mathcal{L}'_{\underline{w}} = \begin{pmatrix} \mathcal{L}'_{w_1} \\ \vdots \\ \mathcal{L}'_{w_n} \end{pmatrix} = 2\Sigma \cdot \underline{w} - \lambda \cdot \underline{u} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

C: $\quad \underline{u}^T \cdot \underline{w} = 1$

$$\boxed{2\Sigma \underline{w} - \lambda \underline{u} = \underline{0}}$$

$$\boxed{\underline{u}^T \cdot \underline{w} = 1}$$

Derivatives of quadratic / linear fcns:

$f(\underline{x}) = \underline{x}^T A \underline{x} \quad \Rightarrow \quad \frac{\partial f}{\partial \underline{x}} = 2A \cdot \underline{x}$

$f(\underline{x}) = \underset{\text{row vector}}{B} \cdot \underline{x} \quad \Rightarrow \quad \frac{\partial f}{\partial \underline{x}} = B^T$

Ex:

$f(\underline{x}) = x^2 + 4xy - y^2$

$= \underline{x}^T \cdot \underbrace{\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}}_{A} \underline{x}$

$\left.\begin{aligned} f'_x &= 2x + 4y \\ f'_y &= 4x - 2y \end{aligned}\right\} \quad f'_{\underline{x}} = \begin{pmatrix} 2x + 4y \\ 4x - 2y \end{pmatrix}$

$= \begin{pmatrix} 2 & 4 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$= 2A \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

$f(\underline{x}) = 3x - 7y$

$= \underbrace{(3\ \ -7)}_{B} \begin{pmatrix} x \\ y \end{pmatrix} \quad f'_{\underline{x}} = \begin{pmatrix} 3 \\ -7 \end{pmatrix} = B^T$

FOC: $2\Sigma \cdot \underline{w} - \lambda \cdot \underline{\mu} = \underline{0} \Rightarrow 2\Sigma \underline{w} = \lambda \cdot \underline{\mu}$

C: $\underline{\mu}^T \cdot \underline{w} = 1$

$\Sigma \underline{w} = \frac{\lambda}{2} \cdot \underline{\mu}$

$\underline{w} = \Sigma^{-1}\left(\frac{\lambda}{2}\underline{\mu}\right) = \frac{\lambda}{2}\left(\Sigma^{-1} \cdot \underline{\mu}\right)$

$\underline{\mu}^T \cdot \frac{\lambda}{2}\left(\Sigma^{-1} \cdot \underline{\mu}\right) = 1$

$\frac{\lambda}{2} \cdot \left(\underline{\mu}^T \Sigma^{-1} \underline{\mu}\right) = 1$

$\frac{\lambda}{2} = \frac{1}{\underline{\mu}^T \cdot \Sigma^{-1} \underline{\mu}}$

$\lambda = \frac{2}{\underline{\mu}^T \cdot \Sigma^{-1} \cdot \underline{\mu}}$

$= \frac{1}{\underline{\mu}^T \cdot \Sigma^{-1} \underline{\mu}} \cdot \Sigma^{-1} \cdot \underline{\mu}$

$\underline{w}^* = \frac{\Sigma^{-1} \cdot \underline{\mu}}{\underline{\mu}^T \Sigma^{-1} \cdot \underline{\mu}}$ $\qquad \lambda^* = \frac{2}{\underline{\mu}^T \Sigma^{-1} \underline{\mu}}$

Use SOC: $h(\underline{w}) = L(\underline{w}; \lambda^*) = \underline{w}^T \Sigma \underline{w} - \lambda^*(\underline{\mu}^T \cdot \underline{w})$

$H(h) = 2\Sigma$ pos. defn. $\Rightarrow$ h convex

$\Rightarrow \underline{w}^* = \frac{\Sigma^{-1} \cdot \underline{\mu}}{\underline{\mu}^T \cdot \Sigma^{-1} \cdot \underline{\mu}}$ minimal variance portfolio.

Ex 1 (see notes)

three assets $(n=3)$

A : Microsoft
B : Nordstrom
C : Starbucks

monthly returns

$$\underline{\mu} = \begin{pmatrix} 0.0427 \\ 0.0015 \\ 0.0285 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 0.1000^2 & 0.0018 & 0.0011 \\ 0.0018 & 0.1044^2 & 0.0026 \\ 0.0011 & 0.0026 & 0.1411^2 \end{pmatrix}$$