Monday 20/09 at 17-1945 in A1-040

Plenary Session I: I will go through (selected problems) from Lecture 1-4

You can propose problems, I will put them in the lecture plan.

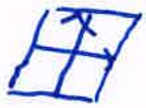
Plan

1 Determinants
2 Minors and rank

Review:

* Vector space: $V = \text{span}(\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}) = \{ c_1 \cdot \underline{v_1} + \dots + c_n \cdot \underline{v_n} : c_1, \dots, c_n \text{ in } \mathbb{R} \}$
* Base of V: A minimal set of vectors that spans V (minimal = linear independent)
* Dimension of V: dim V = the no. of vectors in a base of V

dim V = 1: line   dim V = 2: plane



* Linear independence: $x_1 \underline{v_1} + x_2 \underline{v_2} + \dots + x_n \underline{v_n} = \underline{0}$ has only the trivial solution $x_1 = x_2 = \dots = x_n = 0$ (no free variables)

* Important vector spaces:

$A = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n})$, $m \times n$ matrix

i) $\text{Col}(A) = \text{span}(\underline{v_1}, \dots, \underline{v_n}) \subseteq \mathbb{R}^m$ (column space)
dim Col(A) = rk(A)
Base: pivot columns

ii) $\text{Null}(A) = \{ \underline{x} : A\underline{x} = \underline{0} \} \subseteq \mathbb{R}^n$ (null space)
all solutions $\underline{x}$ of the homogeneous linear system with augmented matrix $(A | \underline{0})$
free variables → dim Null(A) = n − rk(A)
Base: use Gaussian elim.

iii) $\text{Row}(A) = \text{span}(\underline{w_1}, \underline{w_2}, \dots, \underline{w_m}) \subseteq \mathbb{R}^n$
when $\underline{w_1}, \underline{w_2}, \dots, \underline{w_m}$ are the row vectors of A.
dim Row(A) = rk(A)
Base: Row vectors that are non-zero in an echelon form of A

(1) Determinants

$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \rightsquigarrow \det(A) = |A| = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$

$n \times n$ matrix (square)

gives a number $|A| \in \mathbb{R}$

Methods for computing determinants:

i) Cofactor expansion

$C_{11} = + \begin{vmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ (cofactors) $C_{14} = - \begin{vmatrix} 0 & 2 & 3 \\ 0 & 3 & 2 \\ 4 & 0 & 0 \end{vmatrix}$

Ex: $\begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 4 & 0 & 0 & 1 \end{vmatrix} = +1 \cdot \begin{vmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} - 4 \cdot \begin{vmatrix} 0 & 2 & 3 \\ 0 & 3 & 2 \\ 4 & 0 & 0 \end{vmatrix}$

$= +1\left(+1 \cdot \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}\right) - 4\left(+4 \cdot \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}\right)$

$= (1-16) \cdot \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -15(2 \cdot 2 - 3 \cdot 3) = -15 \cdot (-5)$
$= \underline{\underline{75}}$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Facts:
- cofactor expansion along any row or column gives the same result, det(A).
- general method (works for all determinants)
- if A is <u>upper triangular</u>, then det(A) is the product of the diagonal entries.

Ex: $\begin{vmatrix} 1 & 3 & 7 \\ 0 & 7 & -1 \\ 0 & 0 & 3 \end{vmatrix} = 1 \cdot 7 \cdot 3 = 21$ $\quad +1 \cdot \begin{vmatrix} 7 & -1 \\ 0 & 3 \end{vmatrix} = +1 \cdot (7 \cdot 3 - 0)$
$= 1 \cdot 7 \cdot 3 = 21$

ii) Using Gauss:

Ex: $\begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 4 & 0 & 0 & 1 \end{vmatrix} \; -4 = \begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & -15 \end{vmatrix} \; -3/2 =$

$$\begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & -5/2 & 0 \\ 0 & 0 & 0 & -15 \end{vmatrix} = 1 \cdot 2 \cdot (-5/2) \cdot (-15) = \underline{\underline{75}}$$

Fact: $A \to B$ elementary row operation:

i) Switch two rows: $|B| = -|A|$

ii) Multiply a row by $c \neq 0$: $|B| = c \cdot |A|$

iii) Add a multiple of a row to another row: $|B| = |A|$

Consequences: A $n \times n$-matrix

i) $|A| = 0 \iff \text{rk } A < n$

$|A| \neq 0 \iff \text{rk}(A) = n$

ii) When A has two equal rows, or a row that is a scalar multiple of another row (this also works for columns) $\Rightarrow |A| = 0$

$$-1\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = 0 = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \qquad -2\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix} =$$

Summary of results:

$\underset{n \times n \text{ matrix}}{A} = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n})$ $\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}$ column vectors of A

The following statements are equivalent:

i) $|A| \neq 0$

ii) $rk(A) = n$

iii) $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\}$ are linearly independent

iv) A is invertible (A^{-1} exists)

v) $A\underline{x} = \underline{b}$ has one solution for any vector $\underline{b}$

$(A | \underline{b})$

We also have:

(with rows instead of columns)

$|A| \neq 0$

$\Updownarrow$

$rk(A) = n$

$\Updownarrow$

The row vectors of A are linearly independent

2 Minors and rank

A
m×n matrix

Defn: An r-minor of is the determinant of an r×r-submatrix of A.

Ex: $A = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 1 & 2 \end{pmatrix}$

2×3 matrix

r=2: 2-minors = maximal minors

Row 1,2
Col 1,2 → $M_{12,12} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = \underline{3}$

Row 1,2
Col. 2,3 → $M_{12,23} = \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = \underline{0}$

$M_{12,13} = \begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} = \underline{6}$

1-minors: r=1

$M_{1,1} = 1$

$M_{2,1} = -1$

⋮

Result: The rank of A is the maximal order of a ~~max~~ non-zero minor.

r = order ("size" of determinant)

$$\boxed{rk(A) < r \iff \text{all } r\text{-minors are zero}}$$

Ex: $rk\begin{pmatrix} 1 & 2 & 4 \\ -1 & 1 & 2 \end{pmatrix} = \underline{\underline{2}}$ since there is a 2-minor that is non-zero

↓

$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \end{pmatrix}$

Ex:
$$\begin{aligned} x+y+z+w &= 5 \\ 2x-y+3z+4w &= 2 \\ 3x \qquad +4z+6w &= -1 \end{aligned}$$

$$A = \begin{pmatrix} \boxed{\begin{matrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 0 & 4 \end{matrix}} & \begin{matrix} 1 \\ 4 \\ 6 \end{matrix} \end{pmatrix}$$

3×4

Find rk(A) using minors:

When A is square:

$|A| \neq 0 \iff$ there is one solution

$\underline{x} = A^{-1} \cdot \underline{b}$

Max minors = 3-minors:

$$M_{123,123} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 0 & 4 \end{vmatrix} = +3 \cdot (3+1) + 4(-1-2)$$
$$= 12 - 12 = 0$$

$$M_{123,124} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ 3 & 0 & 6 \end{vmatrix} = 3(4+1) + 6 \cdot (-1-2) = 15 - 18 = -3 \neq 0$$

$\Downarrow$

Conclusions:

i) $\text{rk}\, A = 3$

ii) There is one free variable, infinitely many solutions

iii) z is free

$$(A|b) = \left(\begin{array}{c|c} \overbrace{\begin{matrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 4 \\ 3 & 0 & 4 & 6 \end{matrix}}^{A} & \begin{matrix} 5 \\ 2 \\ -1 \end{matrix} \end{array} \right)$$

$$\begin{aligned} x+y+w &= 5 - z = 5-t \\ 2x-y+4w &= 2 - 3z = 2-3t \\ 3x \qquad +6w &= -1 - 4z = -1-4t \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ 3 & 0 & 6 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 5-z \\ 2-3z \\ -1-4z \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} = \frac{1}{-3} \begin{pmatrix} -6 & -6 & 5 \\ 0 & 3 & -2 \\ 3 & 3 & -3 \end{pmatrix} \begin{pmatrix} 5-z \\ 2-3z \\ -1-4z \end{pmatrix}$$
$$= \frac{1}{3} \begin{pmatrix} 47 - 4z \\ z - 8 \\ -24 \end{pmatrix}$$

$z = t$ is free (parameter):

$$(x,y,z,w) = \left(47/3 - 4t/3,\ t/3 - 8/3,\ t,\ -8 \right)$$

Plan

1 Matrix algebra and powers of matrices
2 Properties of determinants

① Matrix algebra and powers of matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$m \times n$ matrix

a_{ij} : entry in pos. (i,j), i.e row i, col. j, in the matrix A

Operations:

i) Addition/subtraction: $A \pm B$ when A, B have the same size

ii) Scalar multiplication: $r \cdot A$

Ex: $\begin{pmatrix} 1 & 2 & 4 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 2 & 1 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 5 \\ 1 & -1 & 4 \end{pmatrix}$

$2 \cdot \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 2 & -2 \end{pmatrix}$

iii) Transpose: $\underset{m \times n}{A} \rightsquigarrow \underset{n \times m}{A^T}$

A is called symmetric if $A^T = A$

Ex: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ not symmetric

$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ symmetric

$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

iv) Special matrices:

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ identity matrix

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ diagonal matrix

$U = \begin{pmatrix} 1 & 7 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ upper triangular

v) Matrix multiplication: $A \cdot B$, with A of size $m \times n$ and B of size $n \times p$

- defined if # cols in A = # rows in B
- result: $m \times p$ matrix
- take inner product of rows in A and columns in B

Ex:

$$\underset{2\times 2}{\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}} \cdot \underset{2\times 2}{\begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} 4 & 4 \\ 1 & 0 \end{pmatrix}$$

$(1,2)\cdot(2,1) = 1\cdot 2 + 2\cdot 1 = \underline{4}$

Properties:

i) $AB \neq BA$

ii) $A\cdot(B+C) = AB + AC$

iii) $A(B\cdot C) = (AB)\cdot C$

iv) $A\cdot I = I\cdot A = A$

v) $(AB)^T = B^T \cdot A^T$

vi) Inverse matrices:

Defn: A $n\times n$ matrix. The inverse of A is a matrix A^{-1} such that

$$A\cdot A^{-1} = I$$

and

$$A^{-1}\cdot A = I$$

Facts:

i) If A is square and $|A| \neq 0$, then A^{-1} exists (A is invertible)

and

$$A^{-1} = \frac{1}{|A|}\cdot \operatorname{adj}(A) = \frac{1}{|A|}\cdot \underbrace{\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}}_{\text{cofactor matrix of } A}{}^T$$

ii) Otherwise, there is no inverse matrix A^{-1}

vii) Matrix form of linear system:

$$\boxed{\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{matrix}}$$

$m\times n$ linear system $\Longleftrightarrow$ $\boxed{A\cdot \underline{x} = \underline{b}}$ matrix form of the linear system

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

In case A is invertible:

(one solution)

$$A \cdot \underline{x} = \underline{b} \quad | A^{-1} \cdot$$

$$A^{-1}(A\underline{x}) = A^{-1}\underline{b}$$

$$\boxed{\underline{x} = A^{-1}\underline{b}}$$

Powers of matrices: $A^2, A^3, A^4, \ldots$

(defined when A is square)

$$A^2 = A \cdot A, \quad A^3 = A^2 \cdot A, \quad A^4 = A^3 \cdot A$$

$m \times n$ $m \times n$

$m = n$

Ex: $\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} = \underline{\begin{pmatrix} 7 & 2 \\ 3 & 6 \end{pmatrix}}$

2×2 2×2

$$\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}^3 = \begin{pmatrix} 7 & 2 \\ 3 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} = \underline{\begin{pmatrix} 13 & 14 \\ 21 & 6 \end{pmatrix}}$$

$\vdots$

$$\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}^n = ?$$

hard for big values of n

need methods for general n

Fact: $D = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_n \end{pmatrix} \Rightarrow D^m = \begin{pmatrix} d_1^m & & 0 \\ & d_2^m & \\ 0 & & \ddots \; d_n^m \end{pmatrix}$

Ex: $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1^2 & 0 \\ 0 & 3^2 \end{pmatrix} = \underline{\begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}}$

$$\begin{pmatrix} 1 & a \\ b & 3 \end{pmatrix}^2 = \begin{pmatrix} 1 & a \\ b & 3 \end{pmatrix}\begin{pmatrix} 1 & a \\ b & 3 \end{pmatrix} = \begin{pmatrix} 1+ab & \underline{4a} \\ \underline{4b} & ab+9 \end{pmatrix}$$

2 Properties of the determinant

Facts:

$$\boxed{\begin{array}{ll} \text{i)} & |AB| = |A|\cdot|B| \\ \text{ii)} & |A^T| = |A| \end{array}}$$

$|A^n| = |A|^n$

$|AB|$

Ex: $\left|\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}\cdot\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\right| = \left|\begin{pmatrix} 4 & 1 \\ 5 & -4 \end{pmatrix}\right| = -16-5 = \underline{-21}$

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$\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}\cdot\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = (-1-6)\cdot(2+1)$

$= -7\cdot 3 = \underline{-21}$

$|A|\cdot|B|$