Plan

1. Linear systems and their geometry
2. Gaussian elimination
3. Rank of a matrix
4. Homogeneous linear systems

Useful additional resources:
[E] Ch. 1
Forklōs: Lecture 1

① Linear systems and their geometry

Ex: $x + y + z + w = 5$
$2x - 3y + z - w = 0$
$x + 6y + 2z + 4w = h$

3×4 linear system with one parameter h

A linear equation in $x_1, x_2, \dots, x_n$ is an equation that can be written

$$a_1x_1 + a_2x_2 + \dots + a_nx = b$$

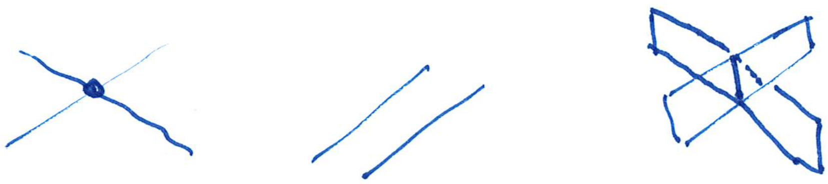
where $a_1, a_2, \dots, a_n$ and b are given numbers.

Geometry: $n=2$ $ax+by=c$ a line in $\mathbb{R}^2$ = xy-plane

$n=3$ $ax+by+cz=d$ a plane in $\mathbb{R}^3$ = xyz-coord. system

In general: $\boxed{a_1x_1 + a_2x_2 + \dots + a_nx_n = b}$ a hyperplane in the non-degenerate case

$a_1 = a_2 = \dots = a_n = 0$: degenerate
all other cases : non-degenerate

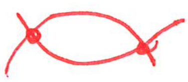


Result A: Any linear system has either

i) no solutions } inconsistent

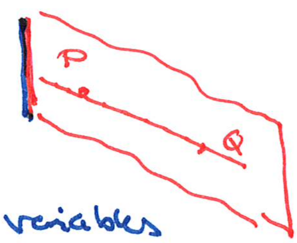
ii) one (unique) solution

iii) infinitely many solutions } the system is consistent



Cannot have this situation in a linear system

Result B "Linear property": If P, Q are two different points on a non-degenerate hyperplane, then all points on the line through P and Q are also on the hyperplane.



Defn: An $m \times n$ linear system in the variables $x_1, \ldots, x_n$ is a system of m linear equations in these variables, which can be written

$$
m \text{ eqn's} \left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ \underbrace{a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n}_{n \text{ variables}} = b_m \end{array} \right. \quad (*)
$$

where a_{ij}, b_i are given numbers.

Alternative notations: $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

coefficient matrix

Representations of the linear system (*):

$(A \mid \underline{b})$ augmented matrix of (*)

$A \cdot \underline{x} = \underline{b}$ matrix form of (*)

② Gaussian elimination

- general method for solving any linear system
- quick - educational

Ex:
$$\begin{aligned} x+y+z+w &= 5 \\ 2x-3y+z-w &= 0 \\ x+6y+2z+4w &= h \end{aligned}$$

aug. matrix
$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 2 & -3 & 1 & -1 & 0 \\ 1 & 6 & 2 & 4 & h \end{array}\right) \begin{array}{l} \\ \leftarrow -2 \\ \leftarrow -1 \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -5 & -1 & -3 & -10 \\ 0 & 5 & 1 & 3 & h-5 \end{array}\right) \begin{array}{l} \\ \\ \leftarrow 1 \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -5 & -1 & -3 & -10 \\ 0 & 0 & 0 & 0 & \boxed{h-15} \end{array}\right)$$

echelon form

Elementary row operations:

i) switch two rows
ii) multiply a row by $c \neq 0$
iii) add a multiple of one row to another row

Defn: Two matrices are row equivalent if you can get from one to the other using a sequence of elementary row operations

Echelon form:

The first nonzero element in a row is called a pivot

A matrix is in echelon form if:

(a) all zero rows are in the bottom
(b) each pivot should be further to the right compared to pivots above

$$x + y + z + w = 5$$
$$-5y - z - 3w = -10$$
$$0 = h - 15$$

Back substitution:

solve for leading variables, starting from the bottom, and substituting the variables you have already solved for

$h = 15$ ↙ ↘ $h \neq 15$

$$x + y + z + w = 5$$
$$-5y - z - 3w = -10$$
$$\cancel{0 = 0}$$

$$x + y + z + w = 5$$
$$-5y - z - 3w = -10$$
$$0 = h - 15 \neq 0$$

no solutions

$$\frac{-5y}{-5} = \frac{-10 + z + 3w}{-5}$$

$$y = 2 - \frac{1}{5}z - \frac{3}{5}w$$

$$x = 5 - y - z - w$$
$$= 5 - \left(2 - \frac{1}{5}z - \frac{3}{5}w\right) - z - w$$
$$= 3 - \frac{4}{5}z - \frac{2}{5}w$$

⇓

$$x = 3 - \frac{4}{5}z - \frac{2}{5}w$$
$$y = 2 - \frac{1}{5}z - \frac{3}{5}w$$
$$z = z \quad \text{(free)}$$
$$w = w \quad \text{(free)}$$

infinitely many solutions

Note:

1. If there is a pivot in the last column, then the system is inconsistent

2. Otherwise, we define a variable to be

   dependent (basic) if it is a pivot column

   free otherwise

no free variables: one unique solution

at least one free variable: infinitely many solutions

\# free variables = number of degrees of freedom

Result: dimension of the set of solutions of $A\underline{x} = \underline{b}$ = \# free variables

③ Rank of a matrix

Let A be any matrix.

Facts: ① A is always row equivalent with an echelon form, but the echelon form is not unique.

② Pivot positions = the position of pivots in an echelon form are unique

Defn: $\operatorname{rk} A := \#$ pivot positions in A

rank of A

Ex:

$$\operatorname{rk} \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -3 & 1 & -1 \\ 1 & 6 & 2 & 4 \end{pmatrix}}_{\text{coeff. matrix}} = 2 \qquad \operatorname{rk} \underbrace{\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 2 & -3 & 1 & -1 & 0 \\ 1 & 6 & 2 & 4 & h \end{array}\right)}_{\text{augmented matrix}} = \begin{cases} 2, & h = 15 \\ 3, & h \neq 15 \end{cases}$$

Theorem

Consider an $m \times n$ linear system with coefficient matrix A and augmented matrix $(A|\underline{b})$. Then we have:

i) If $\operatorname{rk} A \neq \operatorname{rk}(A|\underline{b})$, the system is inconsistent

ii) If $\operatorname{rk} A = \operatorname{rk}(A|\underline{b})$, then the system is consistent with $n - \operatorname{rk}(A)$ degrees of freedom

④ Homogeneous linear systems

The linear system $A\underline{x} = \underline{b}$ is called homogeneous if $\underline{b} = \underline{0}$; that is $b_1 = b_2 = \dots = b_m = 0$.

Ex:

$$\begin{aligned} 3x + 4y + 5z &= 0 \\ 2x + 6y - z &= 0 \\ 4x \qquad + 7z &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 3 & 4 & 5 & 0 \\ 2 & 6 & -1 & 0 \\ 4 & 0 & 7 & 0 \end{array}\right) \cdot(-1) \to \left(\begin{array}{ccc|c} 1 & -2 & 6 & 0 \\ 2 & 6 & -1 & 0 \\ 4 & 0 & 7 & 0 \end{array}\right) -2 \quad -4$$

$$\to \left(\begin{array}{ccc|c} 1 & -2 & 6 & 0 \\ 0 & 10 & -13 & 0 \\ 0 & 8 & -17 & 0 \end{array}\right) \cdot(-1) \to \left(\begin{array}{ccc|c} 1 & -2 & 6 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 8 & -17 & 0 \end{array}\right) -4$$

$$\to \left(\begin{array}{ccc|c} 1 & -2 & 6 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & -33 & 0 \end{array}\right)$$

echelon form

All variables are basic (none of them are free) $\Rightarrow (x, y, z) = \underline{\underline{(0,0,0)}}$

$n - \text{rk}(A) = 3 - 3 = 0$

Note: All homogeneous systems are consistent, and have the trivial solution $\underline{x} = \underline{0}$ or $(x, y, z) = (0, 0, 0)$.

Result: An $m \times n$ homogeneous linear system $A \cdot \underline{x} = \underline{0}$ has non-trivial solutions $\Leftrightarrow$ $\text{rk}\, A < n$

Defn: The nullspace of A is $\text{Null}(A) = \{\underline{x} : A\underline{x} = \underline{0}\}$;

i.e. all the solutions of $A\underline{x} = \underline{0}$.

Fact: $\dim \text{Null}(A) = n - \text{rk}(A)$ when A is $m \times n$ matrix

Ex:

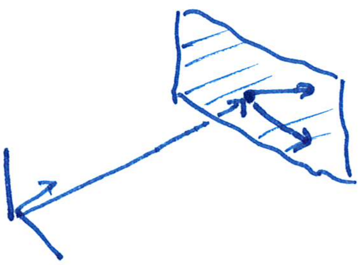
$$\begin{aligned} x+y+z+w &= 5 \\ 2x-3y+z-w &= 0 \\ x+6y+2z+4w &= 15 \end{aligned}$$

inf. many solutions
two degrees of freedom

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 2 & -3 & 1 & -1 & 0 \\ 1 & 6 & 2 & 4 & 15 \end{array}\right) \to \cdots \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -5 & -1 & -3 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{aligned} \underline{x}+y+z+w &= 5 \\ \underline{-5y}-z-3w &= -10 \end{aligned} \implies \begin{aligned} x &= 3 - \tfrac{4}{5}z - \tfrac{2}{5}w \\ y &= 2 - \tfrac{1}{5}z - \tfrac{3}{5}w \\ z, w &: \text{free} \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 - 4/5\,z - 2/5\,w \\ 2 - 1/5\,z - 3/5\,w \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -4/5 \\ -1/5 \\ 1 \\ 0 \end{pmatrix} + w \cdot \begin{pmatrix} -2/5 \\ -3/5 \\ 0 \\ 1 \end{pmatrix}$$



What if the system was homogeneous?

$$\left.\begin{aligned} x+y+z+w &= 0 \\ 2x-3y+z-w &= 0 \\ x+6y+2z+4w &= 0 \end{aligned}\right\} \quad \underline{x} = z \cdot \begin{pmatrix} -4/5 \\ -1/5 \\ 1 \\ 0 \end{pmatrix} + w \cdot \begin{pmatrix} -2/5 \\ -3/5 \\ 0 \\ 1 \end{pmatrix}$$

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To summarize:

$A\underline{x}=\underline{b}$ general linear system

$\Downarrow$

Solution:

$\underline{x} = \begin{Bmatrix} \text{constant} \\ \text{vector} \end{Bmatrix} + t_1 \cdot \underline{w}_1 + \ldots + t_r \cdot \underline{w}_r$

where $t_1, \ldots, t_r$ are the free var's.

$\underline{w}_1, \ldots, \underline{w}_r$ must be computed using Gauss + back substitution, and will be the same vectors as in the homogeneous case

$A\underline{x}=\underline{0}$ homogeneous linear sys (with same matrix A)

$\Downarrow$

Solution:

$\underline{x} = \underline{0} + t_1 \underline{w}_1 + \ldots + t_r \underline{w}_r$

where $t_1, \ldots, t_r$ are the free var's

$\{\underline{w}_1, \ldots, \underline{w}_r\}$ must be computed using Gauss + back substitution

Will talk more next week on bases of vector spaces

$\{\underline{w}_1, \ldots, \underline{w}_r\}$ base of Null(A)