## Plan

1. Determinants
2. Minors and rank
3. Applications to linear systems

**Plenary Session I:**
Mon Sep 19: 16–19
Lecture 1–4

Send suggestions for problems/topics by email.

**Review:** $\{\underline{v}_1, \dots, \underline{v}_n\}$ linearly independent $\iff$ $x_1\underline{v}_1 + \dots + x_n\underline{v}_n = \underline{0}$ has only trivial solution $\underline{x} = \underline{0}$

" linearly dependent $\iff$ — " — has non-trivial solutions

$V$ subset of $\mathbb{R}^m$ is vector space if $V = \text{span}\{\underline{v}_1, \dots, \underline{v}_n\}$ $\iff$ $V$ contains $\underline{0}$ and is closed under linear comb.: $\underline{v}, \underline{w}$ in $V \Rightarrow r\underline{v} + s\underline{w}$ in $V$

A base of $V$ is a minimal set of vectors that spans $V$
$\Updownarrow$
linearly independent — " —

The dimension of $V$ is the number of vectors in a base

$\dim V = 0 \iff V$ is a pt
$\dim V = 1 \iff V$ is a line
$\dim V = 2$: $V$ is a plane

$A$ $m \times n$

i) $V = \text{Col}(A) = \text{span}(\underline{v}_1, \dots, \underline{v}_n)$ in $\mathbb{R}^m$, $A = (\underline{v}_1 | \dots | \underline{v}_n)$ column space

$\dim \text{Col}(A) = \text{rk}(A)$
Base: col. vector corresp. to pivot positions

ii) $V = \text{Null}(A) = \{\underline{x} : A\underline{x} = \underline{0}\}$ in $\mathbb{R}^n$
Solution of homogeneous linear system = null space

$\dim \text{Null}(A) = n - \text{rk}(A) = \#$ free variables
$\text{Null}(A) = \{ t_1 \cdot \underline{w}_1 + t_2 \cdot \underline{w}_2 + \dots + t_r \cdot \underline{w}_r \mid t_1, t_2, \dots, t_r \text{ free vars} \}$
Base $= \{\underline{w}_1, \dots, \underline{w}_r\}$

iii) $V = \text{Row}(A) = \text{span}(\underline{r}_1, \dots, \underline{r}_m)$ in $\mathbb{R}^n$

$$A = \begin{pmatrix} \underline{r}_1 \\ \vdots \\ \underline{r}_m \end{pmatrix} = \begin{pmatrix} \text{—} \ \underline{r}_1 \ \text{—} \\ \text{—} \ \underline{r}_2 \ \text{—} \\ \vdots \end{pmatrix}$$
row space

$\dim \text{Row}(A) = \text{rk}\, A$
Base = {non-zero rows in an echelon form}

## ① Determinants

$A$ ($n \times n$) $\rightsquigarrow$ $\det(A) = |A|$ determinant of $A$, a number

**Ex:** $A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}$

Methods:
i) cofactor expansion
ii) using Gauss

**Method 1:** Cofactor expansion

$$\begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 4 & 0 & 0 & 1 \end{vmatrix} = +1 \cdot \begin{vmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} - 4 \cdot \begin{vmatrix} 0 & 2 & 3 \\ 0 & 3 & 2 \\ 4 & 0 & 0 \end{vmatrix}$$

$$= 1 \cdot \left(+1 \cdot (4-9)\right) - 4\left(+4 \cdot (2 \cdot 2 - 3 \cdot 3)\right) = 1 \cdot 1 \cdot (-5) - 4 \cdot 4(-5)$$

$$= -5 + 80 = \underline{\underline{75}}$$

**Method 2:**

$$|A| = \left|\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}\right| \Big]{\cdot(-4)} = \left|\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & -15 \end{pmatrix}\right| \Big]{\cdot(-3/2)}$$

$$= \left|\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & -5/2 & 0 \\ 0 & 0 & 0 & -15 \end{pmatrix}\right| = |E| = e_{11} \cdot e_{22} \cdot \ldots \cdot e_{nn} = 1 \cdot 2 \cdot (-5/2) \cdot (-15)$$

$e_{11} \cdot e_{22} \cdot \ldots \cdot e_{nn}$: product of diagonal elements

$$= (-5) \cdot (-15) = \underline{\underline{75}}$$

$A \to B$ elementary row operation

- $|A| = |B|$ : add a multiple of one row to another row
- $|A| = -|B|$ : switch two rows $\quad |B| = -|A|$
- $|B| = c \cdot |A|$ : multiply a row by $c \neq 0$

**Note:**

① If $A$ is upper triangular $A = \begin{pmatrix} a_{11} & & * \\ & a_{22} & \\ 0 & & \ddots \; a_{nn} \end{pmatrix}$, then

$|A| = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$.

All echelon forms are upper triangular.

② $\boxed{|A| = 0 \iff rk(A) < n}$ $\iff$ there is not a pivot in every column.

Ex: $A = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \to \ldots \to E = \begin{pmatrix} & & \end{pmatrix}$

$\begin{pmatrix} ① & 2 & 3 \\ 0 & ④ & 7 \\ 0 & 0 & ② \end{pmatrix}$ $rk\, A = n$, $|A| \neq 0$

$\begin{pmatrix} ① & 2 & 3 \\ 0 & 0 & ① \\ 0 & 0 & 0 \end{pmatrix}$ $rk\, A < n$, $|A| = 0$

**Properties of the determinant:**

i) $|A \cdot B| = |A| \cdot |B|$  ii) $|A^T| = |A|$

**Result:** $rk(A)$ = max number of linearly independent columns / rows in $A$ = dim Col(A) / dim Row(A)

**Application:** If one of the columns/rows of $A$ is a linear combination of the others, then $|A| = 0$.

**Ex:**

$$\begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 7 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 5 \end{vmatrix} = 0$$

$R(4) = -R(1)$, $C(4) = -C(1)$  $R(3) = R(1) + R(2)$

**Theorem** Let $A$ be any $n\times n$ matrix. Then the following conditions are equivalent:

i) $|A| \neq 0$

ii) $\text{rk}\, A = n$

iii) the columns of $A$ are linearly independent

iv) the rows of $A$ — " —

v) $A$ is invertible

vi) $A\underline{x} = \underline{b}$ has a unique solution for all $\underline{b}$

there is a pivot position in every column of $A$

The opposite:

$|A| = 0$
⇕
$\text{rk}\, A < n$
⇕
cols linearly dependent
⇕
rows — " —
⇕
$A$ not invertible
⇕
$A\underline{x} = \underline{b}$ has none or inf. many sol'n's

there is at least one column without a pivot position

## ② Minors and rank

$A$ $m\times n$ matrix

**Defn:** An $r$-minor of $A$ (minor of order $r$) is the determinant of an $r\times r$ submatrix of $A$.

$M_{I,J}$ : minor consisting of rows $I$, cols $J$

**Defn** $M_{I,J}$ is a principal minor if $I = J$.

**Ex:** $A = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 1 & 2 \end{pmatrix}$ 2x3 matrix

2-minors = maximal minors

$M_{12,12} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = \underline{3}$ (rows ↑ cols)

$M_{12,23} = \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = \underline{0}$

$M_{12,13} = \begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} = \underline{6}$

1-minors

$M_{1,1} = 1$

$M_{1,2} = 2$

⋮

$M_{2,3} = 2$

Result: $\text{rk}\, A =$ maximal order of a non-zero minor in A

$\Updownarrow$

all r-minors are zero $\Leftrightarrow$ $\text{rk}\, A < r$

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & p & p^2 \end{pmatrix}$

2-minors:

$M_{12,12} = \begin{vmatrix} 1 & 1 \\ 1 & p \end{vmatrix} = p - 1$

$M_{12,23} = \begin{vmatrix} 1 & 1 \\ p & p^2 \end{vmatrix} = p^2 - p$

$M_{12,13} = \begin{vmatrix} 1 & 1 \\ 1 & p^2 \end{vmatrix} = p^2 - 1$

1-minors: $1, 1, 1, 1, p, p^2$

$\text{rk}\, A = 2 \Leftrightarrow$ one of the two-minors are non-zero

$\text{rk}\, A < 2 \Leftrightarrow \begin{cases} M_{12,12} = 0 \\ M_{12,23} = 0 \\ M_{12,13} = 0 \end{cases}$

$\Updownarrow$

$\begin{cases} p - 1 = 0 & p = 1 \\ p^2 - p = 0 & p = 0 \text{ or } p = 1 \\ p^2 - 1 = 0 & p = 1 \text{ or } p = -1 \end{cases} \Leftrightarrow \underline{\underline{p = 1}}$

Conclusion: $\text{rk}\, A = \begin{cases} 2, & p \neq 1 \\ 1, & p = 1 \end{cases}$

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & p & p^2 \\ 1 & q & q^2 \end{pmatrix}$

maximal minors = determinant

$|A| \neq 0 \Leftrightarrow \text{rk} A = 3$

$|A| = 0 \Leftrightarrow \text{rk} A < 3$

$$|A| = 1 \cdot (pq^2 - qp^2) - 1(q^2 - p^2) + 1 \cdot (q - p)$$

$$= pq(q-p) - (q-p)(q+p) + (q-p)$$

$$= (q-p)(pq - p - q + 1) = 0$$

$q - p = 0$ or $pq - p - q + 1 = 0$

$p = q$ $\quad (p-1)(q-1) = 0$

$p = 1$ or $q = 1$

$\text{rk} A = 3$ if $p \neq 1, q \neq 1, p \neq q$

$\text{rk} A < 3$ otherwise

cases: $p = 1, q \neq 1$ $\quad M_{13,12} = q - 1 \neq 0$

$q = 1, p \neq 1$ $\quad M_{12,12} = p - 1 \neq 0$

$p = q \neq 1$ $\quad M_{23,12} = q - 1 \neq 0$

$p = q = 1$ $\quad A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

all 2-minors are zero

$$\Rightarrow \text{rk} A = \begin{cases} 1, & p = q = 1 \\ 2, & p = 1 \neq q \text{ or } q = 1 \neq p \text{ or } p = q \neq 1 \\ 3, & p \neq 1 \text{ and } q \neq 1 \text{ and } p \neq q \end{cases}$$

## (3) Applications to linear systems

Ex:
$$x+y+z+w=5$$
$$2x-y+3z+4w=2$$
$$3x\quad+4z+6w=-1$$

$$A=\begin{pmatrix}1&1&1&1\\2&-1&3&4\\3&0&4&6\end{pmatrix}$$

3-minors: $M_{123,123}=\begin{vmatrix}1&1&1\\2&-1&3\\3&0&4\end{vmatrix}$

$=+3\cdot4+4\cdot(-3)=\underline{0}$

$M_{12,12}=\begin{vmatrix}1&1\\2&-1\end{vmatrix}=-3\neq0$

$M_{123,124}=\begin{vmatrix}1&1&1\\2&-1&4\\3&0&6\end{vmatrix}$

$=+3\cdot5+6\cdot(-3)=-3\neq0$

Conclusion so far:

$\text{rk}\,A=\underline{3}$ since $M_{123,124}\neq0$

one free variable $z$

infinitely many solutions (one degree of freedom)

1 2 4

1 $x+y+w=5-z$

2 $2x-y+4w=2-3z$

3 $3x\quad+6w=-1-4z$

$$\begin{pmatrix}1&1&1\\2&-1&4\\3&0&6\end{pmatrix}\cdot\begin{pmatrix}x\\y\\w\end{pmatrix}=\begin{pmatrix}5-t\\2-3t\\-1-4t\end{pmatrix}$$

$z=t$ (free)

$M_{123,124}\neq0$
$\Rightarrow$ invertible matrix

$$\begin{pmatrix}x\\y\\w\end{pmatrix}=\begin{pmatrix}1&1&1\\2&-1&4\\3&0&6\end{pmatrix}^{-1}\cdot\begin{pmatrix}5-t\\2-3t\\-1-4t\end{pmatrix}$$

$z=t$ (free)

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ 3 & 0 & 6 \end{pmatrix}^{-1} = \frac{1}{-3} \cdot \begin{pmatrix} -6 & 0 & 3 \\ -6 & 3 & 3 \\ 5 & -2 & -3 \end{pmatrix}^T = \frac{1}{3} \begin{pmatrix} 6 & 6 & -5 \\ 0 & -3 & 2 \\ -3 & -3 & 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 6 & 6 & -5 \\ 0 & -3 & 2 \\ -3 & -3 & 3 \end{pmatrix} \cdot \begin{pmatrix} 5-t \\ 2-3t \\ -1-4t \end{pmatrix} \quad \text{and} \quad z = t \text{ (free)}$$

$$= \frac{1}{3} \begin{pmatrix} 47-4t \\ t-8 \\ -24 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 47-4t \\ t-8 \\ 3t \\ -24 \end{pmatrix} = \begin{pmatrix} 47/3 \\ -8/3 \\ 0 \\ -8 \end{pmatrix} + t \cdot \begin{pmatrix} -4/3 \\ 1/3 \\ 1 \\ 0 \end{pmatrix}, \quad t \text{ free}$$