Plan

1 Constrained optimization
2 Lagrange problems
3 Second order conditions

(1) Plenary Session: Monday 16-19
(2) Midterm exam: Fri 09-10
(3) Midterm evaluation

Review:

Unconstrained optimization: $\boxed{\max/\min f(\underline{x})}$

(1) Stationary pts:

$\boxed{f'_{x_1} = f'_{x_2} = \dots = f'_{x_n} = 0}$ (FOC) ← Candidates for max/min

(2) Second derivative test: Local data

$\underline{x}^*$ stationary pt: $H(f)(\underline{x}^*)$ pos def. $\Rightarrow$ $\underline{x}^*$ local min
— " — neg. def. $\Rightarrow$ $\underline{x}^*$ local max
— " — indefinite $\Rightarrow$ $\underline{x}^*$ saddle pt

(3) Concave/convex: Global data

f convex $\Leftrightarrow$ $H(f)(\underline{x})$ pos. semidef. for all $\underline{x}$ $\Rightarrow$ Stationary pts are min
f concave $\Leftrightarrow$ " neg. — " — — " — $\Rightarrow$ — " — are max

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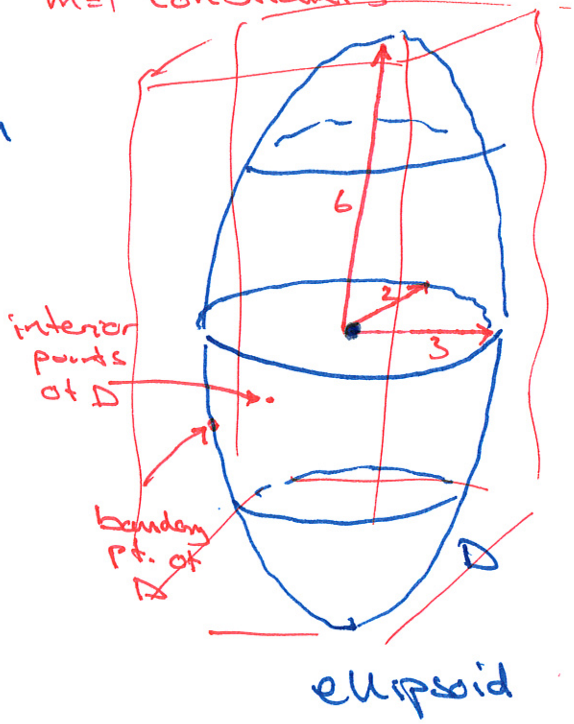
① Constrained optimization

Ex: max/min $\underbrace{f(\underline{x}) = x+y+z}_{\text{obj. fn in } n=3 \text{ variables}}$ when $\underbrace{4x^2+9y^2+z^2=36}_{m=1 \text{ constraints}}$

The set D of <u>adm points</u> in $\mathbb{R}^n$

$$= \{\underline{x} \mid \text{all constraints are satisfied}\}$$

$$= \{(x,y,z) : 4x^2+9y^2+z^2=36\}$$



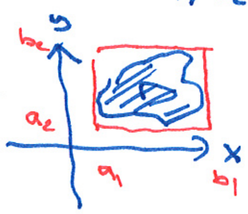
$$\boxed{\begin{array}{l}\max/\min f(\underline{x}) = x+y+z \\ (x,y,z) \text{ in } D\end{array}}$$

If we consider the constraint $4x^2+9y^2+z^2 \leq 36$, then D is a <u>solid</u> ellipsoid.

<u>**Defn:**</u> D is <u>closed</u> if all boundary points of D are included in D and <u>open</u> if none of the — " — are included in D

In concrete terms: $\left\{\begin{array}{ll} = \ \leq \ \geq & \text{closed} \\ < \ > & \text{open}\end{array}\right.$

D is <u>bounded</u> if there are constants $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ such that $\left\{\begin{array}{c} a_1 \leq x_1 \leq b_1 \\ a_2 \leq x_2 \leq b_2 \\ \vdots \\ a_n \leq x_n \leq b_n \end{array}\right\}$ for all $(x_1, \ldots, x_n)$ in D.



$D: 4x^2+9y^2+z^2=36$ is bounded since

$$\left.\begin{matrix} -3 \le x \le 3 \\ -2 \le y \le 2 \\ -6 \le z \le 6 \end{matrix}\right\} \text{for all } (x,y,z) \text{ in } D$$

$D: 4x^2+9y^2+z^2 \le 36$ is bounded

Defn: D is compact if it is closed and bounded.

Theorem (EVT = Extreme Value Thm)
If f is a continuous function defined on a compact set D, then f attains a max and min on D.

(2) Lagrange problems

A constrained optimization problem is called a Lagrange problem if all constraints are equations.

Standard form: Lagrange problem (in n variables with m constraints)

$$\max/\min\ f(x_1,x_2,\dots,x_n)=f(\underline{x}) \quad \text{when} \quad \left\{\begin{matrix} g_1(\underline{x})=a_1 \\ g_2(\underline{x})=a_2 \\ \vdots \\ g_m(\underline{x})=a_m \end{matrix}\right.$$

Expected dimension of $D = n-m$ with $n>m$.

Method of Lagrange multipliers:

$$\mathcal{L}(x_1,x_2,\dots,x_n;\underbrace{\lambda_1,\lambda_2,\dots,\lambda_m}_{\text{Lagrange multipliers}}) = f(\underline{x})-\lambda_1\cdot(g_1(\underline{x})-a_1)-\lambda_2\cdot(g_2(\underline{x})-a_2) - \dots - \lambda_m(g_m(\underline{x})-a_m)$$

Lagrangian function

FOC: $\boxed{L'_{x_1} = L'_{x_2} = \dots = L'_{x_n} = 0}$
C: $\boxed{g_1(\underline{x}) = a_1,\ g_2(\underline{x}) = a_2, \dots, g_m(\underline{x}) = a_m}$

Lagrange conditions (FOC + C)

Idea: Solve FOC+C $\rightsquigarrow$ candidates for max/min

Note: n+m equations in n+m variables
FOC+C $\Leftrightarrow$ Stationary pts of L
since $L'_{\lambda_i} = -(g_i(\underline{x}) - a_i) = 0$

Example: max/min $f(x,y,z) = x+y+z$ when $4x^2+9y^2+z^2 = 36$

$L = x+y+z - \lambda(4x^2+9y^2+z^2-36)$

FOC: $L'_x = 1 - \lambda \cdot 8x = 0$ $\quad 8\lambda x = 1 \quad x = \frac{1}{8\lambda} \quad (\lambda \neq 0)$
$L'_y = 1 - \lambda \cdot 18y = 0$ $\quad 18\lambda y = 1 \quad y = \frac{1}{18\lambda}$
$L'_z = 1 - \lambda \cdot 2z = 0$ $\quad 2\lambda z = 1 \quad z = \frac{1}{2\lambda}$
C: $4x^2+9y^2+z^2 = 36$

$$4\left(\frac{1}{8\lambda}\right)^2 + 9\cdot\left(\frac{1}{18\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 36$$

$$\frac{4}{\underset{8\cdot 8}{64\lambda^2}} + \frac{9}{\underset{18\cdot 18}{18^2\lambda^2}} + \frac{1}{\underset{2\cdot 2}{4\lambda^2}} = 36 \quad \Big| \cdot \lambda^2 \cdot 72^2$$

$$4\cdot\frac{72^2}{8^2} + 9\cdot\frac{72^2}{18^2} + 1\cdot\frac{72^2}{2^2} = 36\cdot 72^2\cdot\lambda^2$$

$$4\cdot 9^2 + 9\cdot 4^2 + 1\cdot 36^2 = 36\cdot 72^2\cdot \lambda^2 \quad \Big| :36$$

$$49 = 9 + 4 + 36 = 72^2 \cdot \lambda^2$$

$$\lambda^2 = \frac{49}{72^2} \qquad \lambda = \pm\frac{\sqrt{49}}{\sqrt{72^2}} = \pm\frac{7}{72}$$

$\lambda = 7/72$: $\frac{1}{\lambda} = \frac{72}{7}$
$x = \frac{1}{8}\cdot\frac{72}{7} = 9/7$
$y = \frac{1}{18}\cdot\frac{72}{7} = 4/7$
$z = \frac{1}{2}\cdot\frac{72}{7} = 36/7$

$\lambda = -7/72$
$x = -9/7$
$y = -4/7$
$z = -36/7$

Candidate points:

$(x,y,z;\lambda) = (9/7, 4/7, 36/7; 7/72)$, $f = 49/7 = 7$ max? Yes.
$(-9/7, -4/7, -36/7; -7/72)$ $f = -49/7 = -7$ min? Yes.

a) EVT: D compact $\Rightarrow$ there is a max and min

b) Necessary conditions for Lagrange problems

Thm: If a Lagrange problem has a max/min $\underline{x}^*$, and the non-degenerate constraint qualification (NDCQ) is satisfied at $\underline{x}^*$, then $(\underline{x}^*; \underline{\lambda}^*)$ satisfies FOC+C for some $\underline{\lambda}^*$.

$\underline{x}^*$ max/min $\underset{\substack{\text{NDCQ} \\ \text{satisfied}}}{\Longrightarrow}$ $(\underline{x}^*, \underline{\lambda}^*)$ satisfies FOC+C

c) NDCQ: $\text{rk}\, J = m$ where $J = \begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 & \cdots & \partial g_1/\partial x_n \\ \vdots & & & \\ \partial g_m/\partial x_1 & \partial g_m/\partial x_2 & \cdots & \partial g_m/\partial x_n \end{pmatrix}$

Ex: $\underbrace{g(x,y,z) = 4x^2 + 9y^2 + z^2}= \underbrace{36}_{a=36}$ $\quad J = \begin{pmatrix} 8x & 18y & 2z \end{pmatrix}$

NDCQ: $\text{rk} \begin{pmatrix} 8x & 18y & 2z \end{pmatrix} = 1$

Adm pts in D where this fails: $8x = 18y = 2z = 0$
$x = y = z = 0$
not admissible

Conclusion: max/min $f = x+y+z$ when $4x^2+9y^2+z^2 = 36$ has $f_{max} = 7$ at $(9/7, 4/7, 36/7)$ with $\lambda = 7/72$
$f_{min} = -7$ " $(-9/7, -4/7, -36/7)$ " $\lambda = -7/72$

Note: If there are adm pts (pts in D, where all constraints are satisfied) where NDCQ fails ($\text{rk}\, J < m$), then these points are also (exceptional) candidate pts.

3 Second order conditions

Thm: (SOC)
If $(\underline{x}^*; \underline{\lambda}^*)$ satisfies FOC+C in a Lagrange problem, then

$h(\underline{x}) = L(\underline{x}, \underline{\lambda}^*)$ convex $\Rightarrow$ $\underline{x}^*$ is min

— " — concave $\Rightarrow$ $\underline{x}^*$ is max

Ex: max/min $f = x+y+z$ when $4x^2+9y^2+z^2 = 36$

$(9/7, 4/7, 36/7; 7/72)$ satisfies FOC+C

$$h(x,y,z) = L(x,y,z; 7/72) = x+y+z - \underline{\tfrac{7}{72}(4x^2+9y^2+z^2-36)}$$

$$H(h) = -\frac{7}{72}\cdot\begin{pmatrix} 8 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -7/9 & 0 & 0 \\ 0 & -7/4 & 0 \\ 0 & 0 & -7/36 \end{pmatrix}$$

$(9/7, 4/7, 36/7)$ is max. $\overset{SOC}{\Longleftarrow}$ h concave $\Leftarrow$ neg. defn. at all pts.

and $f_{max} = f(9/7, 4/7, 36/7) = \underline{\underline{7}}$

Ex: max/min $f(x,y,z,w) = xw - yz$ when $\begin{cases} x^2+4y^2=4 \\ 4z^2+9w^2=36 \end{cases}$

$$\mathcal{L} = xw - yz - \lambda_1(x^2+4y^2-4) - \lambda_2(4z^2+9w^2-36)$$

FOC: $\mathcal{L}'_x = w - \lambda_1 \cdot 2x = 0$ $\quad w = 2\lambda_1 x$

$\mathcal{L}'_y = -z - \lambda_1 \cdot 8y = 0$ $\quad z = -8\lambda_1 y$

$\mathcal{L}'_z = -y - \lambda_2\, 8z = 0$ $\quad -y - 8\lambda_2(-8\lambda_1 y) = 0 \quad -y(1-64\lambda_1\lambda_2) = 0$

$\quad z=0$, $y=0$ or $\lambda_1\lambda_2 = 1/64$

$\mathcal{L}'_w = x - \lambda_2 \cdot 18w = 0$ $\quad x - 18\lambda_2(2\lambda_1 x) = 0 \quad x\cdot(1-36\lambda_1\lambda_2) = 0$

$\quad x=0$, $w=0$ or $\lambda_1\lambda_2 = 1/36$

C: $x^2+4y^2 = 4$

$4z^2+9w^2 = 36$

a) $y=0,\ x=0$: $w=0$ $z=0$ not possible

b) $y=0,\ \lambda_1\lambda_2 = 1/36$: $z=0$ $x^2=4$ $w^2=4$ $\lambda_1 = \frac{w}{2x} = \pm\frac{1}{2}$

$x=\pm 2$ $\quad w = \pm 2$ $\quad \lambda_2 = \pm\frac{1}{18}$

$\Rightarrow (x,y,z,w;\lambda_1,\lambda_2) = (2,0,0,2;\ 1/2, 1/18)$, $f=4$

$(2,0,0,-2;\ -1/2,-1/18)$ $\quad f=-4$

$(-2,0,0,2;\ -1/2,-1/18)$ $\quad f=-4$

$(-2,0,0,-2;\ 1/2,1/18)$ $\quad f=4$

Try SOC on $(2,0,0,2;\ 1/2, 1/18)$:

$$h = xw - yz - \tfrac{1}{2}(x^2+4y^2-4) - \tfrac{1}{18}(4z^2+9w^2-36)$$

$$H(h) = 2A = 2\cdot\begin{pmatrix} -1/2 & 0 & 0 & 1/2 \\ 0 & -2 & -1/2 & 0 \\ 0 & -1/2 & -4/18 & 0 \\ 1/2 & 0 & 0 & -1/2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -4 & -1 & 0 \\ 0 & -1 & -4/9 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

$\underline{\underline{f_{max} = 4}} \overset{\text{SOC}}{\Leftarrow}$ h concave $\Leftarrow$ neg. semidefn. $\Leftarrow$ RRC, rk=3 $\quad D_1 = -1 \quad D_2 = 4 \quad D_3 = -1(16/9 - 1) = -7/9 \quad D_4 = 0$

Try SOC on $(2,0,0,-2;\ -1/2,-1/18)$:

$$h = xw - yz + \frac{1}{2}(x^2+4y^2-4) + \frac{1}{18}(4z^2+9w^2-36)$$

$$H(h) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 4 & -1 & 0 \\ 0 & -1 & 4/9 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \begin{aligned} D_1 &= 1 \\ D_2 &= 4 \\ D_3 &= 1\cdot(16/9-1) = 7/9 \\ D_4 &= 0 \end{aligned}$$

RRC, rk=3 $\Rightarrow$ pos semi $\Downarrow$ h convex $\Downarrow$ SOC

$f_{min} = \underline{\underline{4}}$

Alt: Continue with remaining cases

c) $x=0,\ \lambda_1\lambda_2 = 1/64$: $w=0$, $4y^2=4$, $y^2=1$, $y=\pm1$; $4z^2=36$, $z^2=9$, $z=\pm3$; $\lambda_1 = \frac{z}{-8y} = \pm\frac{3}{8}$, $\lambda_2 = \pm\frac{1}{24}$

$(x,y,z,w;\ \lambda_1,\lambda_2) = (0,1,3,0;\ -3/8,-1/24)$ $f=-3$

$(0,-1,3,0;\ 3/8,1/24)$ $f=3$

$(0,1,-3,0;\ 3/8,1/24)$ $f=3$

$(0,-1,-3,0;\ -3/8,-1/24)$ $f=-3$

d) $\lambda_1\lambda_2 = 1/36 = 1/64$: impossible

Concl: D compact $\begin{pmatrix} -2\le x\le 2 \\ -1\le y\le 1 \\ -3\le z\le 3 \\ -2\le w\le 2 \end{pmatrix} \overset{EVT}{\Rightarrow}$ there is max/min

NDCQ fails: $\text{rk}\begin{pmatrix} 2x & 8y & 0 & 0 \\ 0 & 0 & 8z/9 & 18w \end{pmatrix} < 2 \Rightarrow$ $x=y=0$ or $z=w=0$ not adm.

$\Rightarrow f_{max} = \underline{\underline{4}}$, $f_{min} = \underline{\underline{-4}}$ since this is the max/min f-value among cand. pts.