Plan

1. Kuhn-Tucker problems
2. Second order conditions
3. Envelope theorems

Review:

Constrained optimization $\max/\min\ \underbrace{f(\underline{x})}_{\text{objective function}}$ when $\underline{x}$ satisfies $\Big\}$ list of constraints

$D = \{\underline{x}$: all constraints are satisfied$\}$ adm. pts

① **EVT:** $\left.\begin{array}{l} f \text{ continuous} \\ D \text{ compact (closed and bounded)} \end{array}\right\} \Rightarrow f$ has a max/min in D

② **Lagrange problem:** $\max/\min\ f(\underline{x})$ when $\left\{\begin{array}{l} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{array}\right.$ (equality constraints)

$\underline{x}^*$ is max/min $\underset{\text{NDCQ holds at } \underline{x}^*}{\Longrightarrow}$ $(\underline{x}^*, \underline{\lambda}^*)$ satisfies $\left|\begin{array}{ll} \text{FOC} & L'_{x_1} = L'_{x_2} = \dots = L'_{x_n} = 0 \\ + & \\ \text{C} & g_1(\underline{x}) = a_1, \dots, g_m(\underline{x}) = a_m \end{array}\right.$

for some $\underline{\lambda}^* = (\lambda_1^*, \dots, \lambda_m^*)$, where

$L = f(\underline{x}) - \lambda_1 \cdot (g_1(\underline{x}) - a_1) - \dots - \lambda_m (g_m(\underline{x}) - a_m)$

NDCQ: $\operatorname{rk}(\partial g_i / \partial x_j) = \operatorname{rk} J = m$

③ **SOC for Lagrange:**

If $(\underline{x}^*, \underline{\lambda}^*)$ satisfies FOC+C, then:

$$\boxed{\begin{array}{lll} h(\underline{x}) = L(\underline{x}; \underline{\lambda}^*) & \text{convex} & \Rightarrow \underline{x}^* \text{ min.} \\ \quad -\text{''}- & \text{concave} & \Rightarrow \underline{x}^* \text{ max} \end{array}}$$

① <u>Kuhn-Tucker problems</u> = constrained optimization problem where the constraints are <u>closed</u> <u>inequalities</u> ($\leq$, $\geq$)

<u>Std. form:</u>

$$\max f(\underline{x}) \text{ when } \begin{cases} g_1(\underline{x}) \leq a_1 \\ g_2(\underline{x}) \leq a_2 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{cases}$$

$f(\underline{x}) = f(x_1, \dots, x_n)$

<u>Note:</u> max (not min)
$\leq$ (not $\geq$)

<u>Ex:</u> $\max f(\underline{x}) = x+y+z$ when $4x^2+9y^2+z^2 \leq 36$ KT-prob. (std. form)

<u>Method:</u> Lagrange multipliers

$$L = f(\underline{x}) - \lambda \cdot (g(\underline{x}) - a) = x+y+z - \lambda(4x^2+9y^2+z^2-36)$$

$$\underbrace{\begin{aligned} L'_x &= 1 - \lambda \cdot 8x = 0 \\ L'_y &= 1 - \lambda \cdot 18y = 0 \\ L'_z &= 1 - \lambda \cdot 2z = 0 \end{aligned}}_{\text{FOC}} \qquad \underbrace{4x^2+9y^2+z^2 \leq 36}_{\text{C}} \qquad \underbrace{\begin{aligned} &\lambda \geq 0 \text{ and} \\ &\lambda \cdot (4x^2+9y^2+z^2-36) = 0 \end{aligned}}_{\text{CSC}}$$

CSC (complementary slackness conditions)

<u>Kuhn-Tucker conditions:</u> FOC + C + CSC

Idea: Solve FOC + C + CSC to find candidates for max.

<u>C:</u> i) <u>Binding case</u> $4x^2+9y^2+z^2=36$ <u>CSC:</u> $\lambda \geq 0$
ii) <u>Non-binding case:</u> $4x^2+9y^2+z^2<36$ $\lambda = 0$

i) $4x^2+9y^2+z^2=36$ ($\lambda \ge 0$):

$$\boxed{\begin{array}{l} 1-8\lambda x=0 \\ 1-18\lambda y=0 \\ 1-2\lambda z=0 \end{array}} \qquad \begin{array}{l} x=1/8\lambda \\ y=1/18\lambda \\ z=1/2\lambda \end{array}$$

$$4\cdot\left(\frac{1}{8\lambda}\right)^2+9\left(\frac{1}{18\lambda}\right)^2+\left(\frac{1}{2\lambda}\right)^2=36$$

$$\frac{\cancel{4}}{16\,\cancel{64}\lambda^2}+\frac{\cancel{9}}{2\cdot 18\lambda^2}+\frac{1}{4\lambda^2}=36 \quad \Big|\cdot\lambda^2\cdot 72\cdot 2$$

$$\frac{2\cdot\cancel{72\lambda^2}}{16\cancel{\lambda^2}}+\frac{2\cdot 72\cancel{\lambda^2}}{36\cancel{\lambda^2}}+\frac{2\cdot 72\cancel{\lambda^2}}{4\cancel{\lambda^2}}=\lambda^2\cdot 36\cdot 72\cdot 2$$

$$49=9+4+36=\lambda^2\cdot 72^2$$

$$\lambda^2=\frac{49}{72^2}=\frac{7^2}{72^2}$$

$$\lambda=\pm\frac{7}{72}=\frac{7}{72}$$

$\Rightarrow$

$$x=\frac{72}{7}\cdot\frac{1}{8}=\underline{9/7}$$

$$y=\frac{72}{7}\cdot\frac{1}{18}=\underline{4/7}$$

$$z=\frac{72}{7}\cdot\frac{1}{2}=\underline{36/7}$$

$$\lambda=\underline{7/72}$$

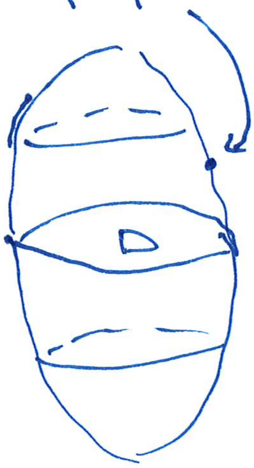
$$f=9/7+4/7+36/7=\underline{7}$$

ii) $4x^2+9y^2+z^2<36$, $\lambda=0$:

FOC: $1=0$ not possible

$\Rightarrow$ no cand. pts in case ii)

Candidate pts: (FOC+C+CSC) $\quad (x,y,z;\lambda)=(9/7,\,4/7,\,36/7;\,7/72) \quad f=7$



ellipsoid

Is this the max?

A) EVT: D is compact (bounded) $\Rightarrow$ there is a max

$\Rightarrow$ assuming NDCQ holds $\quad f_{max}=\underline{\underline{7}}$

B) SOC: $h(x,y,z)=L(x,y,z;7/72)$ concave $\Rightarrow f_{max}=\underline{7}$

General case, KT in std. form:

$$\max f(\underline{x}) \quad \text{when} \quad \begin{cases} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}$$

How to find the std. form

1) $g(\underline{x}) \ge a: \; -g(\underline{x}) \le -a$

2) $\min f(\underline{x}) = \max -f(\underline{x})$

$$L = f(\underline{x}) - \lambda_1 \cdot (g_1(\underline{x}) - a_1) - \ldots - \lambda_m (g_m(\underline{x}) - a_m)$$

FOC: $L'_{x_1} = 0, \ \ldots, \ L'_{x_n} = 0$

C: $g_1(\underline{x}) \le a_1, \ \ldots, \ g_m(\underline{x}) \le a_m$

CSC: $\lambda_1 \ge 0 \quad \lambda_1(g_1(\underline{x}) - a_1) = 0$

$\lambda_2 \ge 0 \quad \lambda_2(g_2(\underline{x}) - a_2) = 0$

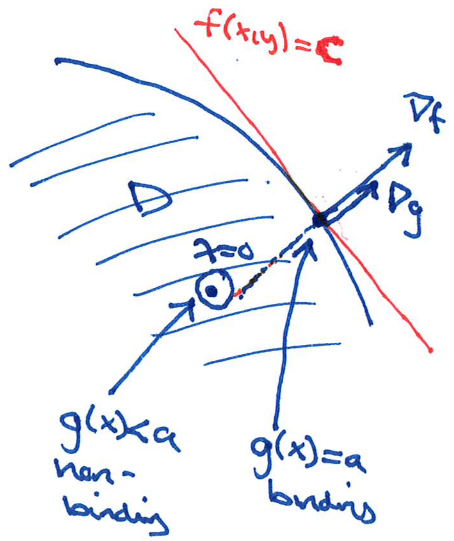
$\vdots$

$\lambda_m \ge 0 \quad \lambda_m(g_m(\underline{x}) - a_m) = 0$

KT conditions: FOC + C + CSC

$$\nabla f = \lambda \cdot \nabla g \qquad \begin{cases} L'_x = f'_x - \lambda g'_x = 0 \\ L'_y = f'_y - \lambda \cdot g'_y = 0 \end{cases}$$

$$\nabla f = \begin{pmatrix} f'_x \\ f'_y \end{pmatrix} \qquad \nabla g = \begin{pmatrix} g'_x \\ g'_y \end{pmatrix}$$



Necessary conditions for maximum:

If $\underline{x}^*$ is a max in a KT problem in std form, and NDCQ holds at $\underline{x}^*$, then $(\underline{x}^*; \underline{\lambda}^*)$ satisfies the KT conditions FOC + C + CSC.

$$\underline{x}^* \text{ max} \underset{\substack{\text{NDCQ} \\ \text{holds at } \underline{x}^*}}{\Longrightarrow} (\underline{x}^*, \underline{\lambda}^*) \text{ satisfies FOC+C+CSC}$$

NDCQ for KT problems

C: $g_1(\underline{x}) \le a_1$
$\vdots$
$g_m(\underline{x}) \le a_m$

NDCQ holds at $\underline{x}^*$ if $\operatorname{rk} J(\underline{x}^*)$ is maximal, where $J(\underline{x}^*)$ consists of those rows in J corresponding to binding constraints at $\underline{x}^*$

Ex. I:

$4x^2+9y^2+z^2 \le 36$

$J = \begin{pmatrix} 8x & 18y & 2z \end{pmatrix}$

i) Binding: $4x^2+9y^2+z^2=36$
$\operatorname{rk}\begin{pmatrix} 8x & 18y & 2z \end{pmatrix} = 1$ ✓ ok

Fails if $x=y=z=0$, not binding.

ii) Nonbinding: $4x^2+9y^2+z^2<36$
no condition ✓ ok

Ex 2:

$x^2+y^2+z^2 \le 4$

$x+y+z \le 3$

$J = \begin{pmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \end{pmatrix}$

i) $x^2+y^2+z^2=4$, $x+y+z=3$: $\operatorname{rk}\begin{pmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \end{pmatrix} = 2$ ✓ ok

ii) $x^2+y^2+z^2=4$, $x+y+z<3$: ~~rk (2x 2y)~~ $\operatorname{rk}\begin{pmatrix} 2x & 2y & 2z \end{pmatrix} = 1$ ✓ ok

iii) $x^2+y^2+z^2<4$, $x+y+z=3$: $\operatorname{rk}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = 1$ ✓ ok

iv) no condition ✓ ok

NDCQ fails: $\operatorname{rk}\begin{pmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \end{pmatrix} < 2 \iff$ all 2-minors are zero

$2x-2y=0 \quad 2y-2z=0$
$\Downarrow \quad 2x-2z=0$

$x=y=z$: $x=y=z=1$
$3 \cdot 1^2 \neq 4$

② Second order condition

SOC: If $(\underline{x}^*; \underline{\lambda}^*)$ is a solution of the KT conditions FOC+C+CSC, then
$h(\underline{x}) = L(\underline{x}; \underline{\lambda}^*)$ concave $\Rightarrow$ $\underline{x}^*$ is max

③ Envelope theorems — Lagrange problems / Kuhn-Tucker problems

Ex: max $f(\underline{x}) = xw - yz$ when $\begin{cases} x^2+4y^2 \le 4 \\ 4z^2+9w^2 \le 36 \end{cases}$ std. form

$L = xw - yz - \lambda_1(x^2+4y^2-4) - \lambda_2(4z^2+9w^2-36)$

FOC:
$L'_x = w - \lambda_1 \cdot 2x = 0$ (1)
$L'_y = -z - \lambda_1 \cdot 8y = 0$
$L'_z = -y - \lambda_2 \cdot 8z = 0$
$L'_w = x - \lambda_2 \cdot 18w = 0$ (4)

C: $x^2+4y^2 \le 4$
$4z^2+9w^2 \le 36$

CSC: $\lambda_1 \ge 0$
$\lambda_1(x^2+4y^2-4) = 0$
$\lambda_2 \ge 0$
$\lambda_2(4z^2+9w^2-36) = 0$

Alt I: $w = 2\lambda_1 x$
$x - 18\lambda_2(2\lambda_1 x) = 0$
$x(1 - 36\lambda_1\lambda_2) = 0$
$x = 0$ or $\lambda_1\lambda_2 = 1/36$

$x=0, w=0$

$\Downarrow$
$\lambda_1, \lambda_2 > 0$
$\Downarrow$
$x^2+4y^2 = 4$
$4z^2+9w^2 = 36$

$z = -8\lambda_1 y$
$-y - 8\lambda_2(-8\lambda_1 y) = 0$
$-y(1 - 64\lambda_1\lambda_2) = 0$
$y = 0$ or $\lambda_1\lambda_2 = 1/64$

$y=0, z=0$

$\lambda_1, \lambda_2 > 0$
$x^2+4y^2 = 4$
$4z^2+9w^2 = 36$

(1)+(4): $x=0, w=0$ or $\lambda_1\lambda_2 = 1/36$ (2)+(3): $y=0, z=0$ or $\lambda_1\lambda_2 = 1/64$

i) $x=0, w=0$ and $y=z=0$: Both constraints non-binding $\Rightarrow \lambda_1=\lambda_2=0$

$(0,0,0,0;0,0)$ $f=0$

ii) $x=0, w=0$ and $\lambda_1\lambda_2 = 1/64$: — " — binding $4y^2=4$, $4z^2=36$

$\Rightarrow y=\pm 1$, $z=\pm 3$

$\lambda_1 = -\frac{1}{8}\frac{z}{y} = 3/8$ y, z opposite signs

$\lambda_2 = 1/24$

$(0,1,-3,0;3/8,1/24)$, $f=3$

$(0,-1,3,0;3/8,1/24)$ $f=3$

iii) $\lambda_1\lambda_2 = 1/36$ and $y=0, z=0$: — " — binding $x^2=4$, $9w^2=36$

$\Rightarrow x=\pm 2$, $w=\pm 2$

$\lambda_1 = \frac{1}{2}\frac{w}{x} = \frac{1}{2}$, w, x same sign

$\lambda_2 = 1/18$

$(2,0,0,2;1/2,1/18)$ $f=4$

$(-2,0,0,-2;1/2,1/18)$ $f=4$

iv) $\lambda_1\lambda_2 = 1/36$ $\lambda_1\lambda_2 = 1/64$: impossible

SOC for $(2,0,0,2;1/2,1/18)$, $(-2,0,0,-2;1/2,1/18)$

$$h = xw - yz - \tfrac{1}{2}(x^2+4y^2-4) - \tfrac{1}{18}(4z^2+9w^2-36)$$

$$= 4 + \underline{x}^T A \underline{x} \quad \text{with}$$

$$A = \begin{pmatrix} -1/2 & 0 & 0 & 1/2 \\ 0 & -2 & -1/2 & 0 \\ 0 & -1/2 & -2/9 & 0 \\ 1/2 & 0 & 0 & -1/2 \end{pmatrix}$$

$D_1 = -1/2$ $D_2 = 1$

$D_3 = -1/2\,(4/9 - 1/4)$

$= -\frac{1}{2}\left(\frac{16}{36} - \frac{9}{36}\right) = -\frac{7}{72} < 0$

$D_4 = 0$ $(R(4) = -R(1))$

$\Downarrow$ RRC

A neg semidefin.

$\Downarrow$

h concave

$f_{max} = 4$ at $\leftarrow$ SOC

$(2,0,0,2), (-2,0,0,-2)$

with $\lambda_1 = 1/2$, $\lambda_2 = 1/18$

⇓

(See complete calc. on previous pages)

$(x,y,z,w;\lambda_1,\lambda_2) = \underline{(2,0,0,2;\ 1/2,\ 1/18)}$, $\quad f=4$

$\underline{(-2,0,0,-2;\ 1/2,\ 1/18)}$, $\quad f=4$

$\Big\}$ max

⋮

③ **Envelope thm:**

$\max\ f(\underline{x}) = xw - yz$ when $\begin{cases} x^2+4y^2 \le 4 \\ 4z^2+9w^2 \le 36 \end{cases}$

has max value $f_{max} = 4$

$\underline{x}^* = (2,0,0,2),\ (-2,0,0,-2)$

$\underline{\lambda}^* = (1/2,\ 1/18)$

What happens if I change the problem?

i) What if we change xw to $2xw$ in f

$\max\ f = axw - yz$ when $\begin{cases} x^2+4y^2 \le 4 \\ 4z^2+9w^2 \le 36 \end{cases}$

$\mathcal{L} = axw - yz - \lambda_1(x^2+4y^2-4) - \lambda_2(4z^2+9w^2-36)$

$\mathcal{L}'_a = xw$

Envelope thm: $\boxed{\dfrac{df^*(a)}{da} = \mathcal{L}'_a(x^*(a);\lambda^*(a))}$

$a=1$:

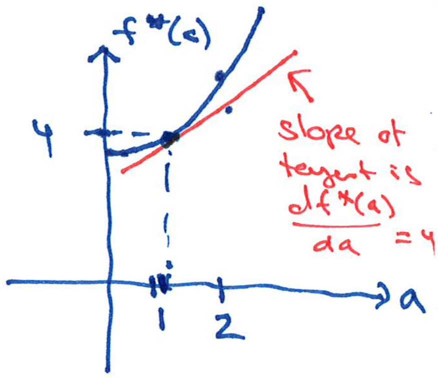
$\underline{x}^*(1) = (2,0,0,2),\ (-2,0,0,-2)$

$\lambda^*(1) = (1/2,\ 1/18)$

$f^*(1) = 4$

$a=2$: ?

$f^*(a)$ is the maximum value fn, the maximum value when a is given



Ex: $\dfrac{df^*(a)}{da} = \mathcal{L}'_a(x^*(\underline{a});\lambda^*(a))$

$= x^*(a)\cdot w^*(a)$

$= x^*(1)\cdot w^*(1) = 4$

$f^*(2) \simeq f^*(1) + (2-1)\cdot 4 = 4 + 1\cdot 4 = \underline{\underline{8}}$

For Lagrange / KT problems: with parameter a

$f^*(a)$ = max value when parameter has value a

Envelope Thm:

$$\boxed{\frac{df^*(a)}{da} = \mathcal{L}'_a(x^*(a);\lambda^*(a))}$$