Complex numbers

12/9-22

Today: 1) Complex numbers and roots of equations.
2) Polar coordinates and De Moivres formula.

1: Complex numbers and roots of eqns.

Ex: $x^2 - 4x + 5 = 0$

$$x^2 - 4x = -5$$

complete the square

$$x^2 - 4x + 4 = -5 + 4$$

$$(x-2)^2 = -1$$

Define: $i := \sqrt{-1}$

$$x - 2 = \pm\sqrt{-1}$$

$$x - 2 = \pm i$$

$$x = 2 \pm i$$

Hence, $x_1 = \underline{2 + i}$ and $x_2 = \underline{2 - i}$

Alternative: $x^2 - 4x + 5 = 0$

$$x = \frac{4 \pm \sqrt{16 - 4\cdot 5}}{2} = \frac{4 \pm \sqrt{-4}}{2}$$

$$= \frac{4 \pm \sqrt{4}\sqrt{-1}}{2}$$

$-4 = 4 \cdot (-1)$

$$= \frac{4 \pm 2\sqrt{-1}}{2}$$

$$= 2 \pm \sqrt{-1}$$

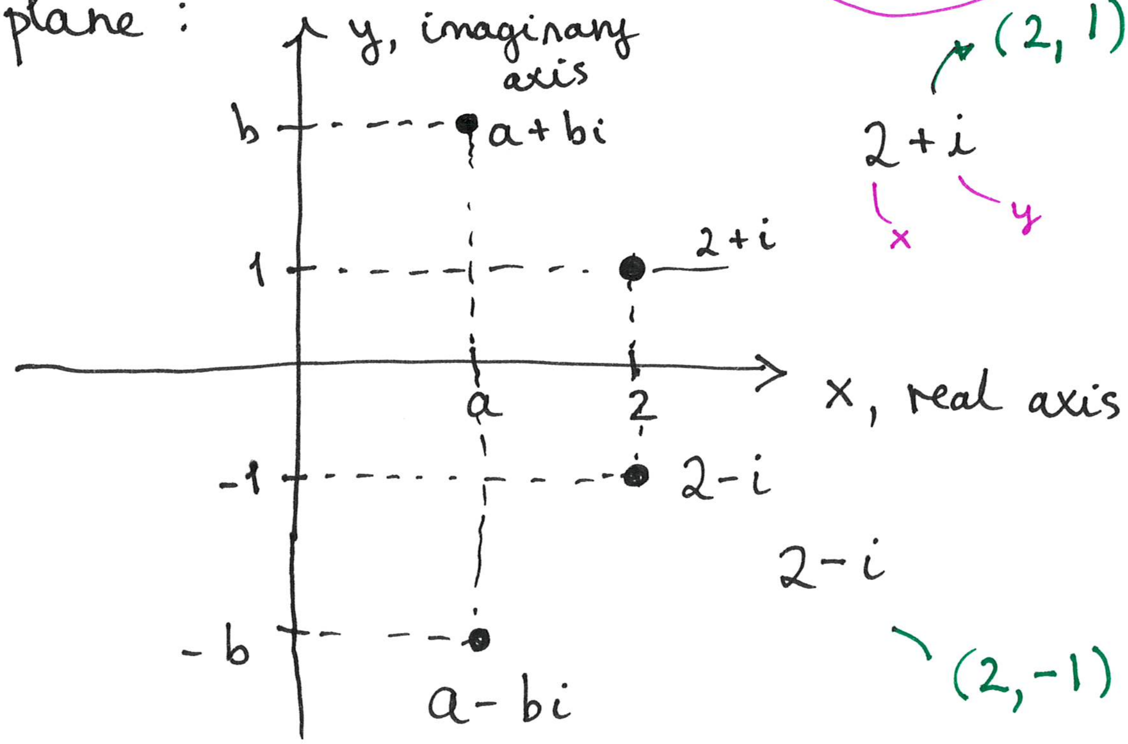
$$= \underline{2 \pm i}$$

Def (complex number): $z = a + bi$, where $a, b \in \mathbb{R}$

a, b are real numbers, $(a, b) \in \mathbb{R}^2$

→ Since $(a, b) \in \mathbb{R}^2$, we can represent complex numbers via points in the plane :

$a + bi$: $a \to x$, $b \to y$



$2 + i \to (2, 1)$: $2 \to x$, $1 \to y$

$2 - i \to (2, -1)$

Ex: $z_1 = 2 + 3i, \quad z_2 = 1 - i$

$$z_1 + z_2 = 2 + 3i + (1 - i) = 2 + 1 + 3i - i$$
$$= \underline{3 + 2i}$$

$$z_1 - z_2 = 2 + 3i - (1 - i) = 2 - 1 + 3i + i$$
$$= \underline{1 + 4i}$$

$$z_1 \cdot z_2 = (2 + 3i)(1 - i) = 2 - 2i + 3i - 3i^2$$
$$= 2 + i - 3(-1)$$
$$= \underline{\underline{5 + i}}$$

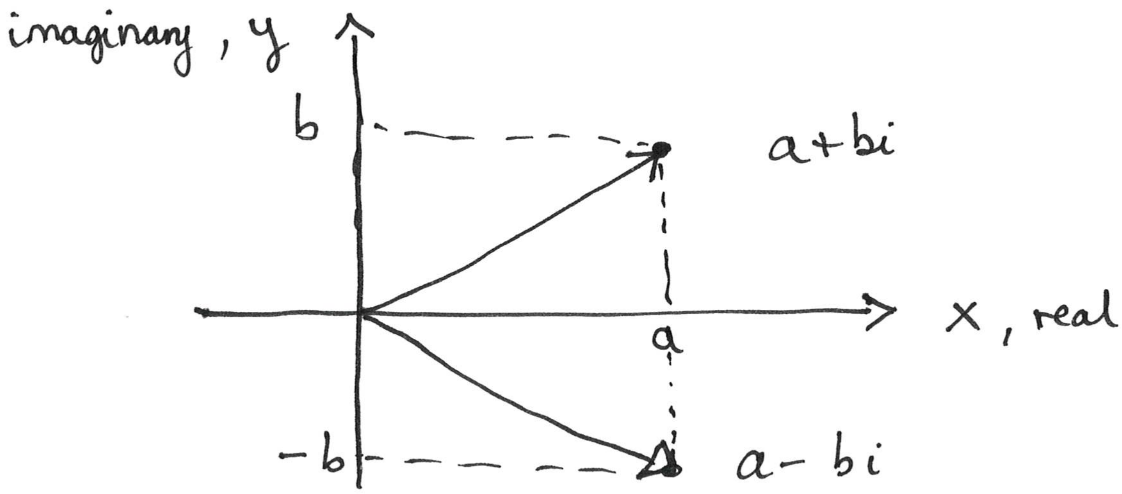
$i = \sqrt{-1}$

$i^2 = -1$

Ex: $(1 + i)(1 - i) = 1 - \cancel{i} + \cancel{i} - i^2 = \underline{\underline{2}}$

Def (complex conjugate): Let $z = a + bi$ be a complex number. The complex conjugate of z, $\bar{z}$, is

$$\bar{z} := a - bi$$

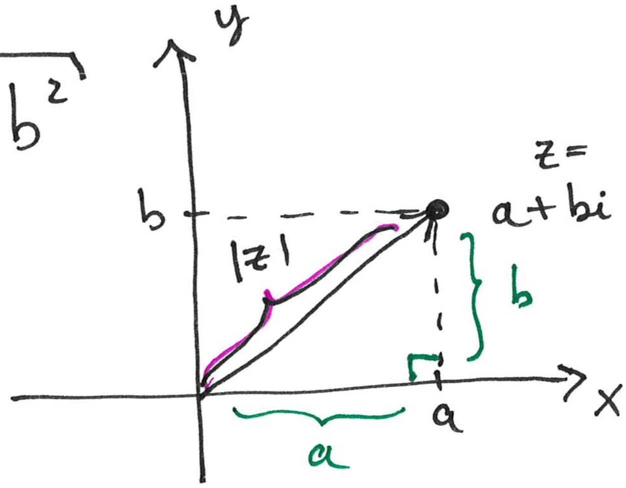


Def (modulus of z):

The modulus of a complex number $z = a+bi$ denoted $|z|$, is

$$|z| := \sqrt{a^2+b^2}$$

The length of the vector (a,b) in $\mathbb{R}^2$, from Pythagoras



Pythagoras: $a^2+b^2 = |z|^2$

$$|z| = \sqrt{a^2+b^2}$$

FORMULA: $z \cdot \bar{z} = |z|^2$

Pf: Let $z = a+bi$. Then, $\bar{z} = a-bi$.

So:
$$\begin{aligned} z\cdot\bar{z} &= (a+bi)\cdot(a-bi) \\ &= a^2 - \cancel{abi} + \cancel{bai} - b^2 \underbrace{i^2}_{-1} \\ &= a^2+b^2 \\ &= |z|^2 \end{aligned}$$

-1 because $i := \sqrt{-1}$

■

Example (division):

$$\frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{\cancel{1}+i+i+\cancel{i^2}}{2} = \frac{2i}{2} = \underline{\underline{i}}$$

IDEA: Expand by complex conjugate of denominator to get real denominator

TRY AT HOME: $\frac{1+i}{2+3i} = \ldots$

$$= \frac{5}{13} - \frac{1}{13}i$$

Roots of equations:

Ex: $x^3 - 5x^2 + 17x - 13 = 0$

Try $x=1$: $1^3 - 5 \cdot 1^2 + 17 \cdot 1 - 13 = 1 - 5 + 17 - 13 = 0$

$\Downarrow$

$x=1$ is a solution.

$\Rightarrow$ $(x-1)$ is a factor in

$i^3 = i^2 i$
$= (-1) i = -i$
$i^4 = i^2 i^2 = (-1)(-1)$
$= 1$

Polynomial division:

$$(x^3 - 5x^2 + 17x - 13) \div (x-1) = x^2 - 4x + 13$$

$$-(x^3 - x^2)$$

$$-4x^2 + 17x - 13$$

$$-(-4x^2 + 4x)$$

$$13x - 13$$

$$-(13x - 13)$$

$$0$$

$$(x-1)(x^2 - 4x + 13) = 0$$

$x = 1$ or $x^2 - 4x + 13 = 0$

$$x = \frac{4 \pm \sqrt{16 - 4 \cdot 13}}{2}$$

$$= \frac{4 \pm \sqrt{4}\sqrt{4 - 13}}{2}$$

$$= \frac{4 \pm 2\sqrt{-9}}{2}$$

$$= \underline{2 \pm 3i}$$

$-9 = (-1) \cdot 3^2$

Roots:

$\underline{x_1 = 1}$, $x_2 = \underline{2 + 3i}$, $x_3 = \underline{2 - 3i}$

$\overline{x_2}$

Thm: Any polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

Fund. thm. of algebra

with $a_0, a_1, \dots, a_n \in \mathbb{R}$ and $a_n \neq 0$ has

i) n complex solutions

ii) If z is a solution, so is $\bar{z}$.

Ex: $x^3 = 1$

$x^3 - 1 = 0$

Try: $x = 1$; $1^3 - 1 = 1 - 1 = 0$

$x = 1$ is a solution!

Polynomial division: $(x-1)$

At home!

$$(x^3 - 1) : (x-1) = x^2 + x + 1$$

2nd order formula

$$(x-1)(x^2 + x + 1) = 0$$

$$x = 1 \quad \text{or} \quad x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$= \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{3}\, i}{2}$$

$-3 = 3(-1)$

$\sqrt{-3} = \sqrt{3}\,\underbrace{\sqrt{-1}}_{i}$

Roots: $x_1 = 1$, $x_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$, $x_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$(= \overline{x_2})$

What is the modulus?

$$z^3 = 1$$

$$|z^3| = |1| = 1$$

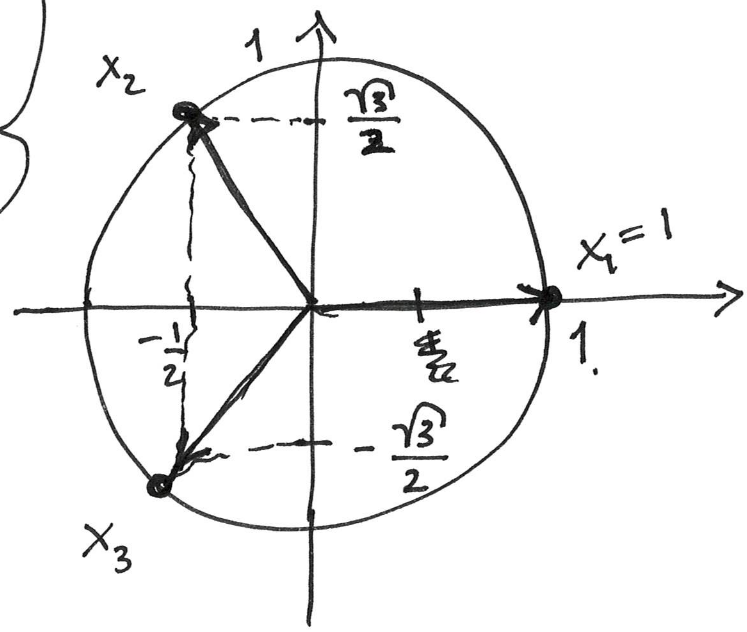
$$|z|^3 = 1, \quad \text{so } |z| = 1$$

Lemma A_2:

$$|z_1 z_2| = |z_1||z_2|$$

$$\Downarrow$$

$$|z^n| = |z|^n$$



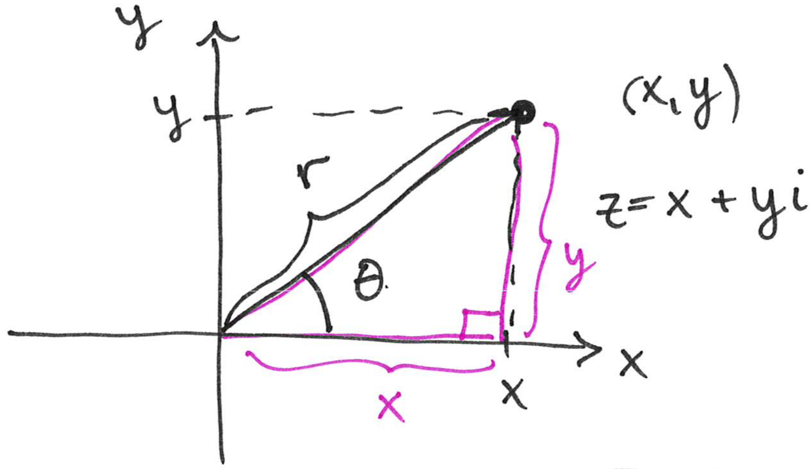
Polar coordinates and de Moivre's formula

(x, y); Cartesian coordinates

(r, θ); polar coordinates

distance to $(0,0)$

angle to pos. x-axis counterclockwise



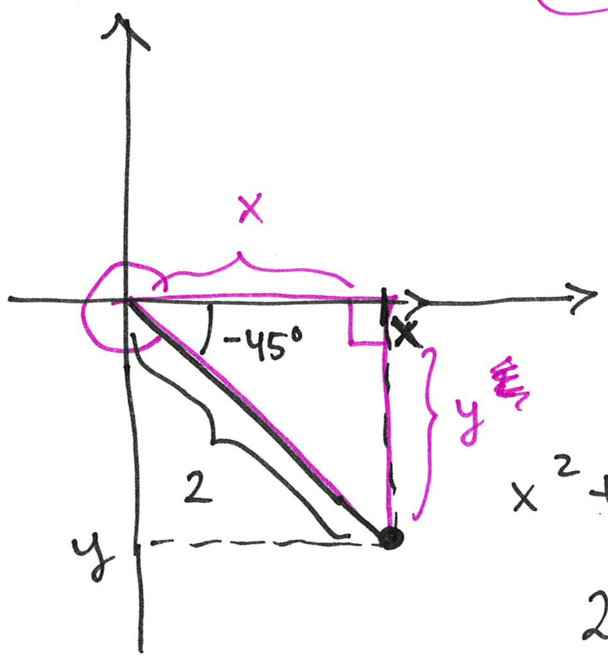
Here; $\quad r = |z| = \sqrt{x^2 + y^2}$

NB: θ is only def. up to integer multiples of 360°.

E. ex: 30° & 390° is the same angle.

Ex: $(r, \theta) = (2, -45°)$

-45° is the same as 360° - 45° = 315°



Because of -45°, $x = |y|$. Hence,

$$x^2 + y^2 = 2^2$$

$$2y^2 = 4$$

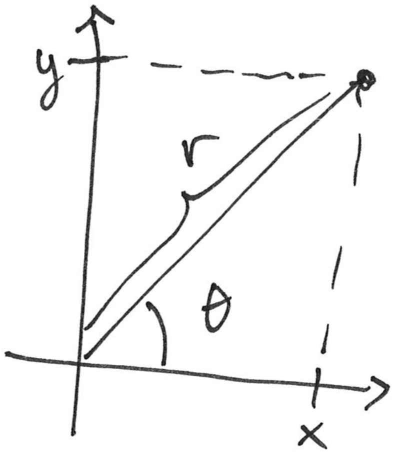
$$y^2 = 2$$

From fig. y is negative, so

$$\underline{y = -\sqrt{2}} \quad \text{and} \quad \underline{x = \sqrt{2}}.$$

So $(r, \theta) = (2, -45°) \Leftrightarrow (x, y) = (\sqrt{2}, -\sqrt{2})$.

NOTE:



$z = x + iy$

Def (polar coordinates):

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

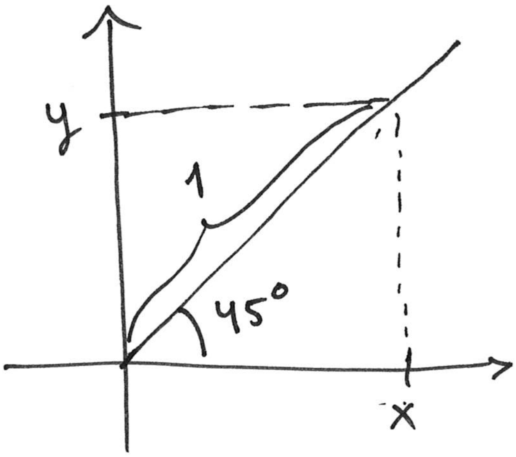
From def of sin & cos

$$\Downarrow$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Ex: sin/cos of 45°?



$x^2 + y^2 = 1$

and (due to 45°) $y = x$.

$$2y^2 = 1$$

$$y^2 = \frac{1}{2}$$

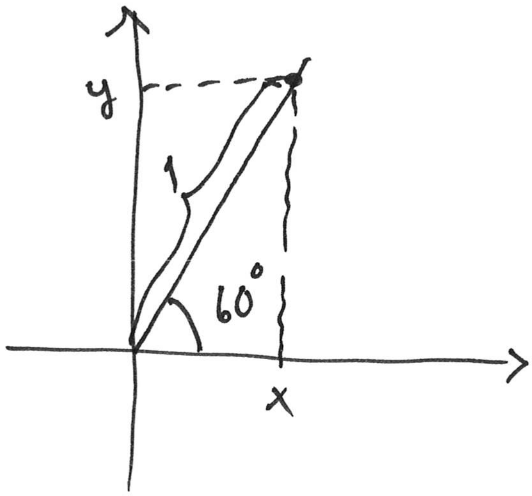
y positive from figure

$$y = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = x$$

Hence: $\sin(45°) = \frac{y}{r} = \frac{1}{\sqrt{2}} \approx 0{,}707$

$$\cos(45°) = \frac{x}{r} = \frac{1}{\sqrt{2}}$$

Ex:
$$\sin(60°) = \frac{\sqrt{3}}{2}$$
$$\cos(60°) = \frac{1}{2}$$



Writing complex numbers via polar coordinates

$$z = x + iy = r\cos\theta + i\, r\sin\theta$$

$x = r\cos\theta$
$y = r\sin\theta$

$$= r(\cos\theta + i\sin\theta)$$

NB: i) $|z| = r$

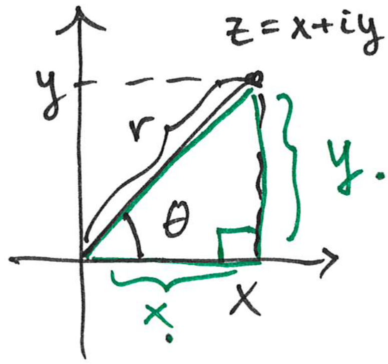
Why? $|z| = |r(\cos\theta + i\sin\theta)|$

$= |r|\,|\cos\theta + i\sin\theta|$

scaling can be put outside

$= r\,\underbrace{|\cos\theta + i\sin\theta|}_{1}$, Why?

$r \geq 0$



$$|\cos\theta + i\sin\theta|^2 = \cos^2\theta + \sin^2\theta$$
$$= \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2$$
$$= \frac{x^2+y^2}{r^2} = \frac{r^2}{r^2} = 1$$

Proposition (Euler's formula)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Thm (de Moivres formula): Let $z = r(\cos\theta + i\sin\theta)$ be a complex number. Then,

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

for any positive integer $n \geq 1$.

Note: $z = re^{i\theta} \Rightarrow z^n = (re^{i\theta})^n = r^n \underbrace{e^{in\theta}}_{(e^{in\theta})}$

Ex: $z^5 = 1$ $\rightarrow$ $z^5 - 1 = 0$, Fundamental thm of algebra: 5 complex roots

In general:

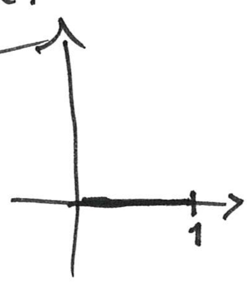
$$z = r(\cos\theta + i\sin\theta)$$

$\Downarrow$ (De Moivre)

$$z^5 = r^5(\cos(5\theta) + i\sin(5\theta))$$

For our eq: Right hand side, 1, in polar coord:

$$1 = 1(\underbrace{\cos 0^\circ}_{1} + i\underbrace{\sin 0^\circ}_{0})$$



$\Downarrow$

$r^5 = 1 \Rightarrow r = 1$

Angles :
$$5\theta = 0^\circ + k\cdot 360^\circ, \quad k \text{ integer}$$
$$\theta = 0^\circ + k\cdot\frac{360^\circ}{5}, \quad k = 0,1,2,3,4$$
$$\theta = 0^\circ + k\cdot 72^\circ, \quad k = 0,1,2,3,4$$
$$\Downarrow$$

k=0: $z_0 = 1\,(\overbrace{\cos 0^\circ}^{1} + i\overbrace{\sin 0^\circ}^{0})$
$= 1$

k=1: $z_1 = 1\,(\cos 72^\circ + i \sin 72^\circ)$
$= \cos 72^\circ + i\sin 72^\circ$

$\vdots$

$k=2$, $k=3$, $k=4$.