| | Plan |
|---|---|
| 1 | Complex vector spaces and Hermitian matrices |
| 2 | Non-negative matrices and Perron-Frobenius theory |
| 3 | Ranking models and Population growth models |

## ① Complex vector spaces and Hermitian matrices

**Ex:** $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad \lambda^2 + 1 = 0$ $\quad$ no real eigenvalues

$\lambda^2 = -1$

$\lambda = \pm\sqrt{-1} = \pm i \quad \lambda_1 = i,\ \lambda_2 = -i$

complex eigenvalues

Eigenvectors:

$\lambda = i$: $\quad E_i = \text{Null}\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$

$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \cdot i \to \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \cdot i \to \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$ $\quad x - iy = 0$, $y$ free

$\underline{v} = \begin{pmatrix} iy \\ y \end{pmatrix} = y \cdot \begin{pmatrix} i \\ 1 \end{pmatrix} \quad \underline{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$\lambda = -i$: $\quad \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \to \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$ $\quad x + iy = 0$, $y$ free

$\underline{v} = \begin{pmatrix} -iy \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -i \\ 1 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

<u>Complex vectors:</u> $\underline{v} = (v_1, v_2, \dots, v_n)$ where $v_i$ is a complex number

<u>Inner product:</u> $\langle \underline{v}, \underline{w} \rangle = \bar{v}_1 w_1 + \bar{v}_2 w_2 + \dots + \bar{v}_n w_n$

$\quad = \underline{v} \cdot \underline{w}$ $\qquad \boxed{\overline{a+bi} = a - bi}$

<u>Ex:</u> $(i,1)\cdot(-i,1) = \bar{i}\cdot(-i) + \bar{1}\cdot 1 = (-i)(-i) + 1\cdot 1$
$= i^2 + 1 = 0$

$(i,1)\cdot(i,1) = \bar{i}\cdot i + \bar{1}\cdot 1 = (-i)i + 1\cdot 1$
$= -i^2 + 1 = 1 + 1 = \underline{2}$

$(i,1)\cdot(i,1) = \|(i,1)\|^2$

<u>Note:</u> $\langle \underline{v}, \underline{v} \rangle = \|\underline{v}\|^2 \geq 0$

$\underline{v} \perp \underline{w} \iff \underline{v}\cdot\underline{w} = 0$

<u>Defn:</u>
$A = (a_{ij})$ complex matrix $\qquad A^* = (\overline{a_{ij}})^T$

<u>Ex:</u> $A = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \qquad A^* = \begin{pmatrix} \overline{-i} & \overline{-1} \\ \bar{1} & \overline{-i} \end{pmatrix}^T = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}^T$

$= \underline{\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}}$

<u>Defn:</u>
$A$ is <u>Hermitian</u> if $A^* = A$ $\qquad$ (real: symmetric
and <u>unitary</u> if $A^* = A^{-1}$. $\qquad$ orthogonal )

<u>Ex:</u> $\begin{pmatrix} 2 & 1-i \\ 1+i & 3 \end{pmatrix}$ is Hermitian

**Theorem:** A Hermitian $n \times n$ matrix

i) A has $n$ eigenvalues that are <u>real</u> (counted with mult.)

ii) If $\underline{v}, \underline{w}$ are eigenvectors of A with distinct eigenvalues, then $\underline{v} \cdot \underline{w} = 0$.

iii) There is a unitary matrix U such that $U^* A U = D$ is diagonal.

<u>Pf:</u>

i) If $\lambda$ is an eigenvalue of A, with eigenvector $\underline{v}$, then

$$\langle A\underline{v}, \underline{v} \rangle = \langle \lambda \underline{v}, \underline{v} \rangle = (\lambda \underline{v})^* \underline{v} = \bar{\lambda}\, \underline{v}^* \underline{v} = \bar{\lambda} \cdot \|\underline{v}\|^2$$

$$\langle v, A\underline{v} \rangle = \langle \underline{v}, \lambda \underline{v} \rangle = \underline{v}^* \lambda \underline{v} = \lambda\, \underline{v}^* \underline{v} = \lambda \cdot \|\underline{v}\|^2$$

$\Downarrow$

$\lambda = \bar{\lambda}$, ie. $\lambda$ is <u>real</u>

<u>Note:</u> i) <u>Fundamental thm of algebra</u>

$a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 = 0$ has $n$ complex roots when $a_n, \dots, a_1, a_0$ are real/complex numbers.

ii) A is Hermitian $\Leftrightarrow$ $\langle A\underline{v}, \underline{w} \rangle = \langle \underline{v}, A\underline{w} \rangle$

$$\langle \underline{v}, \underline{w} \rangle = \bar{v}_1 w_1 + \dots + \bar{v}_n w_n = \underline{v}^* \cdot \underline{w} = (\bar{v}_1 \ \bar{v}_2 \ \dots \ \bar{v}_n) \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

$\Downarrow$

$$\langle A\underline{v}, w \rangle = (A\underline{v})^* \underline{w} = \underline{v}^* A^* \underline{w}$$

$$\langle \underline{v}, A\underline{w} \rangle = \underline{v}^* A \underline{w}$$

ii) $A\underline{v} = \lambda_1 \underline{v}$, $A\underline{w} = \lambda_2 \underline{w}$ $(\lambda_1 \neq \lambda_2)$

$$\langle \underline{v}, A\underline{w} \rangle = \langle \underline{v}, \lambda_2 \underline{w} \rangle = \lambda_2 \langle \underline{v}, \underline{w} \rangle$$

$$\langle A\underline{v}, \underline{w} \rangle = \langle \lambda_1 \underline{v}, \underline{w} \rangle = \bar{\lambda}_1 \langle v, w \rangle = \lambda_1 \langle \underline{v}, \underline{w} \rangle$$

$\Downarrow$

$$(\lambda_2 - \lambda_1) \langle v, w \rangle = 0 \Rightarrow \langle \underline{v}, \underline{w} \rangle = 0$$

iii) A Hermitian: Let $\underline{v}_1$ be an eigenvector of $A$, with eigenvalue $\lambda_1$

Defn: $V_2 = \{ \underline{w} : \langle \underline{v}_1, \underline{w} \rangle = 0 \} = \text{Null}(\underline{v}_1^*)$
$\underline{v}_1^* \cdot \underline{w} = 0$ (subspace of dim $n-1$)

Note: $v_2$ in $V_2 \Rightarrow A\underline{v}_2$ in $V_2$

$\underline{v}_1^* \cdot \underline{v}_2 = 0$

$\langle \underline{v}_1, \underline{v}_2 \rangle = 0$

$$\begin{aligned} \langle \underline{v}_1, A\underline{v}_2 \rangle &= \langle A\underline{v}_1, \underline{w}_2 \rangle \\ &= \langle \lambda_1 \underline{v}_1, \underline{v}_2 \rangle \\ &= \overline{\lambda_1} \langle v_1, v_2 \rangle \\ &= \lambda_1 \langle \underline{v}_1, \underline{v}_2 \rangle = 0 \end{aligned}$$

$$\begin{aligned} A : V_2 &\to V_2 \\ \underline{v}_2 &\mapsto A \cdot \underline{v}_2 \end{aligned}$$ Hermitian

induction

You find eigenvectors $\underline{v}_1, \underline{v}_2, \underline{v}_3, \dots, \underline{v}_n$ of $A$ which are orthogonal.

$$U = \left( \frac{1}{\|v_1\|} \underline{v}_1 \,\middle|\, \frac{1}{\|v_2\|} \underline{v}_2 \,\middle|\, \dots \right) \qquad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{pmatrix}$$

# Untitled9

September 28, 2020

[1]:
```
import numpy as np
from numpy import linalg as LA
```
✓

[2]:
```
A = np.array([[0.75,0.02,0.1],[0.2,0.9,0.2],[0.05,0.08,0.7]])
B = np.array([[0,0,6],[1/2,0,0],[0,1/3,0]])
```

[3]:
```
LA.eig(A)
```

[3]:
```
(array([1.  , 0.7 , 0.65]),
 array([[-1.88144174e-01, -8.08122036e-01, -7.07106781e-01],
        [-9.40720868e-01,  5.05076272e-01, -1.59384710e-15],
        [-2.82216261e-01,  3.03045763e-01,  7.07106781e-01]]))
```
eig. values

base of $E_1$

$\lambda=0.7$ $\lambda=0.65$

[4]:
```
LA.matrix_power(A,5)
```
$A^5$

[4]:
```
array([[0.28940219, 0.09010763, 0.17337312],
       [0.55462   , 0.72269   , 0.55462   ],
       [0.15597781, 0.18720238, 0.27200687]])
```

[5]:
```
LA.matrix_power(A,10)
```

[5]:
```
array([[0.16077148, 0.12365308, 0.14730873],
       [0.64783498, 0.67608251, 0.64783498],
       [0.19139354, 0.20026441, 0.20485628]])
```

[6]:
```
LA.matrix_power(A,100)
```
$\underline{u}$ eq. state

[6]:
```
array([[0.13333333, 0.13333333, 0.13333333],
       [0.66666667, 0.66666667, 0.66666667],
       [0.2       , 0.2       , 0.2       ]])
```

[7]:
```
LA.eig(B)
```

[7]:
```
(array([-0.5+0.8660254j, -0.5-0.8660254j,  1. +0.j       ]),
 array([[ 0.88465174+0.j        ,  0.88465174-0.j        ,
         -0.88465174+0.j        ],
        [-0.22116293-0.38306544j, -0.22116293+0.38306544j,
         -0.44232587+0.j        ],
```
$j=\sqrt{-1}=i$

```
        [-0.07372098+0.12768848j, -0.07372098-0.12768848j,
         -0.14744196+0.j         ]]))
```

```
[8]: LA.matrix_power(B,5)
```

```
[8]: array([[0.        , 2.        , 0.        ],
            [0.        , 0.        , 3.        ],
            [0.16666667, 0.        , 0.        ]])
```

```
[9]: LA.matrix_power(B,10)
```

```
[9]: array([[0.        , 0.        , 6.        ],
            [0.5       , 0.        , 0.        ],
            [0.        , 0.33333333, 0.        ]])
```

```
[10]: LA.matrix_power(B,100)
```

k=3

```
[10]: array([[0.        , 0.        , 6.        ],
             [0.5       , 0.        , 0.        ],
             [0.        , 0.33333333, 0.        ]])
```

```
[11]: LA.matrix_power(B,101)
```

```
[11]: array([[0.        , 2.        , 0.        ],
             [0.        , 0.        , 3.        ],
             [0.16666667, 0.        , 0.        ]])
```

```
[12]: LA.matrix_power(B,102)
```

```
[12]: array([[1., 0., 0.],
             [0., 1., 0.],
             [0., 0., 1.]])
```

```
[ ]:
```

## (2) Non-negative matrices and Perron-Frobenius th.

$A = (a_{ij})$ $n \times n$-matrix

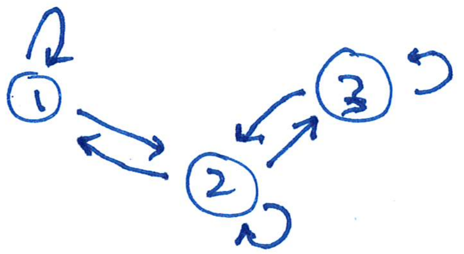
$\underline{v} = (v_i)$ $n$-vector

**Interpretation:** Graph $\Gamma$ of $A$

nodes: $1, 2, 3, \ldots, n$

edges: $i \to j$ if $a_{ji} > 0$

Ex: $A = \begin{pmatrix} 0.8 & 0.1 & 0 \\ 0.2 & 0.7 & 0.3 \\ 0 & 0.2 & 0.7 \end{pmatrix} \geq 0$



strongly connected

A irreducible

**Fact:**

A positive $\Rightarrow$ A primitive $\Rightarrow$ A irreducible

**Defn:** A is positive $A > 0$ if $a_{ij} > 0$ for all $i, j$

A is non-negative $A \geq 0$ if $a_{ij} \geq 0$ — " —

**Defn:**

The graph is strongly connected if there is a path from any node to any other node.

A non-negative matrix $A \geq 0$ is irreducible if the graph is strongly connected, and A is primitive if $A^m > 0$ for some positive integer $m \geq 1$.

A primitive $\Leftrightarrow$ there exist an integer $m \geq 1$ such that you can get from any node to any other node in exactly $m$ steps.

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ irreducible, not primitive


$A^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

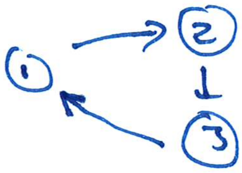
$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Note: For Markov chains, regular $\overset{!}{=}$ primitive.

Ex: $A = \begin{pmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{pmatrix}$.

irreducible, not primitive




$$\begin{vmatrix} -\lambda & 0 & 6 \\ 1/2 & -\lambda & 0 \\ 0 & 1/3 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 0) + 6 \cdot \left(\tfrac{1}{6} - 0\right) = 0$$

$$-\lambda^3 + 1 = 0$$

$$\underline{\lambda^3 = 1} \qquad \underline{\lambda_1 = 1} \quad \text{or} \quad \underline{\lambda^2 + \lambda + 1 = 0}$$

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}\,i}{2}$$

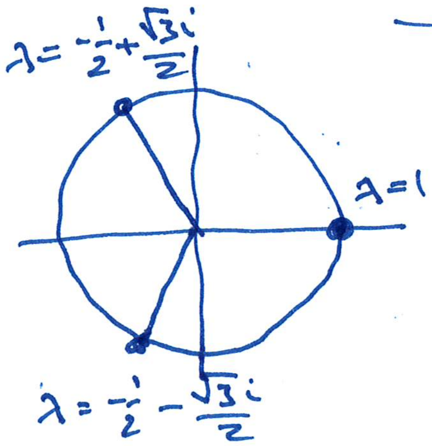
$(\lambda^3 - 1) : (\lambda - 1) = \underline{\lambda^2 + \lambda + 1}$

$\underline{\lambda^3 - \lambda^2}$

$\lambda^2 - 1$

$\underline{\lambda^2 - \lambda}$

$\lambda - 1$



**Thm** (Perron - Frobenius)

Let $A \geq 0$ $n \times n$ irreducible matrix. Then:

i) There is a <u>dominant eigenvalue</u> $\lambda_A > 0$ such that $|\lambda| \leq \lambda_A$ for all other (real or complex) eigenvalues of $A$.

ii) There is a <u>positive</u> eigenvector $\underline{v}_A > 0$ with eigenvalue $\lambda_A$

iii) $\lambda_A$ has multiplicity 1, and any eigenvector $\underline{v} \geq 0$ of $A$ is a positive multiple of $\underline{v}_A$.

iv) all eigenvalues of $A$ with $|\lambda| = \lambda_A$ are all the solutions of $\lambda^k = \lambda_A^k$ for some $k \geq 1$ (index of primitivity)

**Note:** $k=1$ if $A$ is primitive
$k>1$ otherwise

**Ex:** $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

not primitive

$\lambda^2 - 1 = 0$ $\quad \underline{\lambda_1 = 1}$ $\quad \underline{\lambda_2 = -1}$


$\underline{\lambda_A = 1}$: $\lambda^2 = 1$

$k = 2$ $\quad \underline{v}_A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{E_1}$: $\begin{pmatrix} -1 & 1 \\ \cancel{1} & \cancel{-1} \end{pmatrix}$

$\underline{v} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{E_{-1}}$: $\begin{pmatrix} 1 & 1 \\ \cancel{1} & \cancel{1} \end{pmatrix}$

$y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$A = \begin{pmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{pmatrix}$

$\lambda = 1$ or $\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ $\quad$ not primitive

$\underline{\lambda_A = 1}$: $\quad \underline{v_A}$: $\begin{pmatrix} -1 & 0 & 6 \\ 1/2 & -1 & 0 \\ 0 & 1/3 & -1 \end{pmatrix}$

$k = 3$ $\quad \to \begin{pmatrix} -1 & 0 & 6 \\ 0 & 1/3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$\underline{v} = \begin{pmatrix} 6z \\ 3z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}$ $\qquad$ $-x + 6z = 0$
$\frac{1}{3}y - z = 0$
$z$ free

$\underline{v}_A = \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}$

# 3) Population growth models

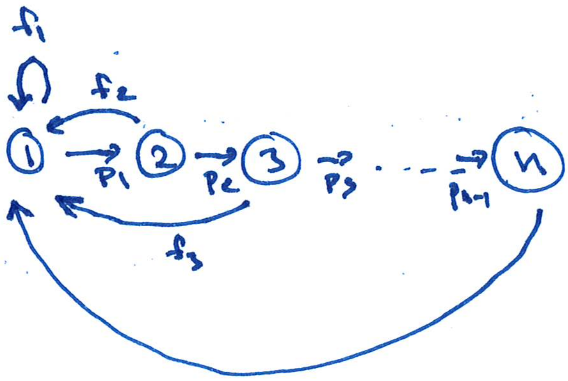
## Leslie matrices:

$$A = \begin{pmatrix} f_1 & f_2 & f_3 & \cdots & f_{n-1} & f_n \\ p_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-1} & 0 \end{pmatrix}$$

$n \times n$ matrix where

$f_1, \dots, f_n \geq 0 \quad f_n \neq 0$

$0 < p_i < 1$




$x_1$: 0 – 1 years old

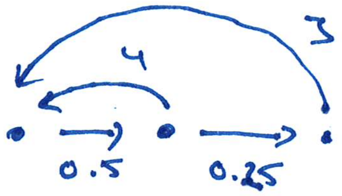
$x_2$: 1 – 2 — " —

$\vdots$

$x_n$: $(n-1) - n$ — " —

A is irreducible

A is primitive $\Leftrightarrow$ more than one $f_i \neq 0$




Ex: $A = \begin{pmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{pmatrix}$

$$\begin{vmatrix} -\lambda & 4 & 3 \\ 0.5 & -\lambda & 0 \\ 0 & 0.25 & -\lambda \end{vmatrix} = 0 \qquad -\lambda(\lambda^2 - 2) - 0.25(-1.5) = 0$$

$$-\lambda^3 + 2\lambda + 0.375 = 0$$

i) Use Python to find eigenvalues / eigenvector

ii) Compute $A^m \cdot (\underline{v_0})$

$\underline{v_0} = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix} \quad \underline{v_1} = A\underline{v_0} = \begin{pmatrix} 70 \\ 5 \\ 2.5 \end{pmatrix} \quad \underline{v_2} = A\underline{v_1} = \begin{pmatrix} 27.5 \\ 35 \\ 0.625 \end{pmatrix}$

# Example Leslie matrices

October 6, 2020

[8]:
```python
import numpy as np
import pandas as pd
```

[9]:
```python
A = np.array([[0,4,3],[0.5,0,0],[0,0.25,0]])
v = np.array([[10],[10],[10]])
```

[3]:
```python
eval, evect = np.linalg.eig(A)
```

[4]:
```python
eval
```

[4]:
```
array([ 1.5       , -1.30901699, -0.19098301])
```

[5]:
```python
w = evect[:,0]
```

[7]:
```python
w/w.sum()
```

[7]:
```
array([0.72, 0.24, 0.04])
```

[10]:
```python
rows = v.shape[0]
for i in range(12):
    last = v.shape[1]
    w = np.reshape(v[:,last-1],(rows,1))
    v = np.append(v,A.dot(w),axis=1)
```

[11]:
```python
v
```

[11]:
```
array([[1.00000000e+01, 7.00000000e+01, 2.75000000e+01, 1.43750000e+02,
        8.12500000e+01, 2.97812500e+02, 2.16406250e+02, 6.26093750e+02,
        5.44492188e+02, 1.33333984e+03, 1.32376953e+03, 2.87086426e+03,
        3.14754150e+03],
       [1.00000000e+01, 5.00000000e+00, 3.50000000e+01, 1.37500000e+01,
        7.18750000e+01, 4.06250000e+01, 1.48906250e+02, 1.08203125e+02,
        3.13046875e+02, 2.72246094e+02, 6.66669922e+02, 6.61884766e+02,
        1.43543213e+03],
       [1.00000000e+01, 2.50000000e+00, 1.25000000e+00, 8.75000000e+00,
        3.43750000e+00, 1.79687500e+01, 1.01562500e+01, 3.72265625e+01,
        2.70507812e+01, 7.82617188e+01, 6.80615234e+01, 1.66667480e+02,
```

```
            1.65471191e+02]])
```

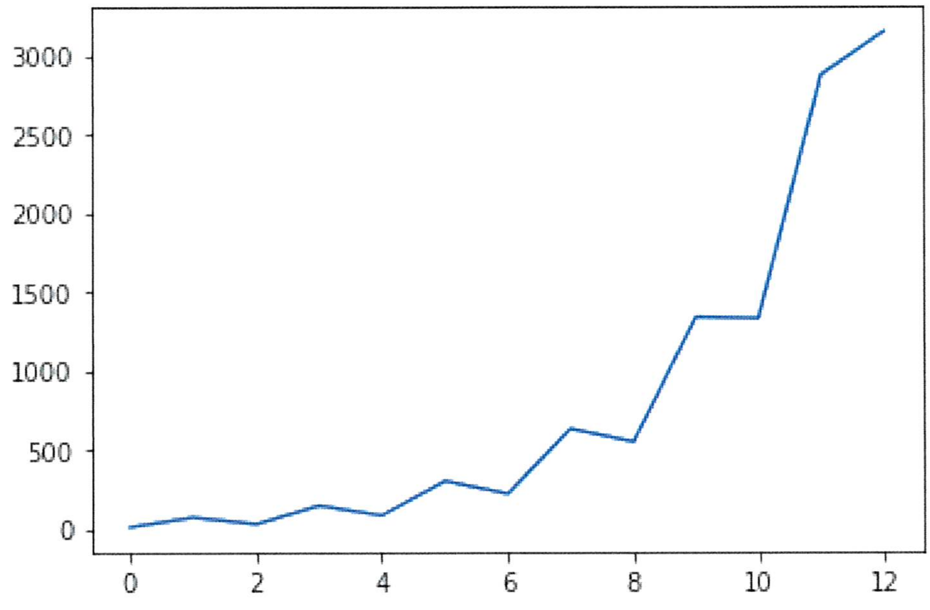
[12]:
```
u = pd.Series(v[0])
```

[13]:
```
u
```

[13]:
```
0       10.000000
1       70.000000
2       27.500000
3      143.750000
4       81.250000
5      297.812500
6      216.406250
7      626.093750
8      544.492188
9     1333.339844
10    1323.769531
11    2870.864258
12    3147.541504
dtype: float64
```

[14]:
```
u.plot()
```

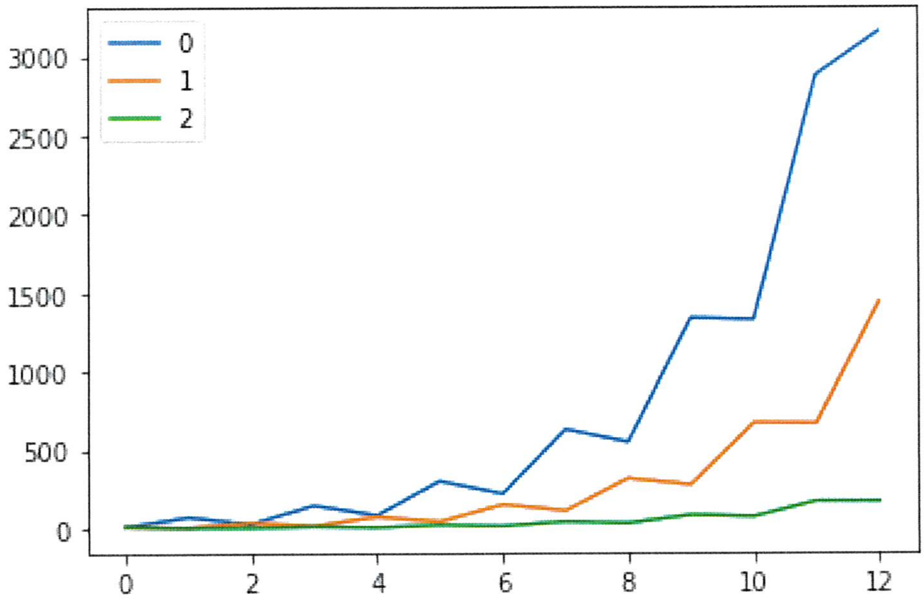
[14]: <matplotlib.axes._subplots.AxesSubplot at 0x26ca0cf7490>




[17]:
```
u = pd.DataFrame(np.transpose(v))
```

[18]:
```
u.plot()
```

[18]: <matplotlib.axes._subplots.AxesSubplot at 0x26ca0ed6250>




[ ]: