	Plan
1	Financial time series in Python
2	Minimum variance portofolis

① Financial time series in Python

t	x_t	r_t (continuous return at time t)
0	x_0	
1	x_1	$\ln(x_1/x_0) = r_1$
2	x_2	$\ln(x_2/x_1) = r_2$
3	x_3	$\ln(x_3/x_2) = r_3$
$\vdots$		$\vdots$
$n-1$	x_{n-1}	$\ln(x_{n-1}/x_{n-2}) = r_{n-1}$
n	x_n	$\ln(x_n/x_{n-1}) = r_n$

$$r_t = \ln(x_t/x_{t-1}) : \quad e^{r_t} = x_t/x_{t-1} \implies \underline{x_t = x_{t-1} \cdot e^{r_t}}$$

$$\bar{r} = \frac{1}{n}(r_1 + r_2 + \dots + r_n) = \frac{1}{n} \cdot \left[\ln(x_1/x_0) + \ln(x_2/x_1) + \dots + \ln(x_n/x_{n-1})\right]$$

$$= \frac{1}{n} \ln\left(x_n/x_0\right) = \underline{\ln\left(x_n/x_0\right)^{1/n}}$$

← CAGR (compounded aggregate growth rate)
– continuous time

2) **Minimum variance portfolios:**

Problem A: $\min f(\underline{w}) = \underline{w}^T \Sigma \underline{w}$ when $\underline{e}^T \cdot \underline{w} = 1$

$f(\underline{w}) = \text{Var}(R)$, $\quad \underline{e}^T \cdot \underline{w} = w_1 + \dots + w_n = 1$

$$L = \underline{w}^T \Sigma \underline{w} - \lambda(\underline{e}^T \underline{w} - 1) \qquad \Sigma, \underline{e}$$

FOC: $L'(\underline{w}) = 2\Sigma \cdot \underline{w} - \lambda(\underline{e}) = \underline{0}$

C: $\underline{e}^T \cdot \underline{w} = 1$

FOC $\Sigma \cdot \underline{w} = \frac{\lambda}{2} \underline{e}$ $\qquad \Sigma$ pos. defn. $\Rightarrow |\Sigma| > 0$

$\Sigma^{-1}\Sigma \cdot \underline{w} = \Sigma^{-1} \cdot \frac{\lambda}{2} \underline{e}$

$\underline{w} = \frac{\lambda}{2} \Sigma^{-1} \underline{e}$

C $\underline{e}^T \cdot \underline{w} = 1$

$\underline{e}^T \cdot \left(\frac{\lambda}{2} \Sigma^{-1} \underline{e}\right) = 1$

$\frac{\lambda}{2} \left(\underline{e}^T \Sigma^{-1} \underline{e}\right) = 1$

Σ pos. defn. $(\lambda_1, \dots, \lambda_n > 0)$

$\Rightarrow \Sigma^{-1}$ pos. defn. $(1/\lambda_1, 1/\lambda_2, \dots, 1/\lambda_n > 0)$ positive

$\Rightarrow \underline{e}^T \cdot \Sigma^{-1} \underline{e} > 0$

$$\underline{\lambda = \frac{2}{\underline{e}^T \Sigma^{-1} \underline{e}} > 0}$$

$$\underline{\underline{w} = \frac{1}{\underline{e}^T \Sigma^{-1} \underline{e}} \cdot \Sigma^{-1} \underline{e}}$$

SOC:

$h = \underline{w}^T \Sigma \underline{w} - \frac{2}{\underline{e}^T \Sigma^{-1} \underline{e}} (\underline{e}^T \underline{w} - 1)$

$H(h) = 2\Sigma$ pos. defn.

$\Downarrow$

h convex

$\Downarrow$ SOC

minimum

Conclusion:

$$\boxed{\underline{w}_M = \frac{1}{\underline{e}^T \Sigma^{-1} \underline{e}} \cdot \Sigma^{-1} \underline{e}}$$

<u>Problem B:</u> $\min f(\underline{w}) = \underline{w}^T \Sigma \underline{w}$ wrt $\begin{cases} \underline{e}^T \cdot \underline{w} = 1 \\ \underline{\mu}^T \cdot \underline{w} = \alpha \end{cases}$

(α given)

$$L = \underline{w}^T \Sigma \underline{w} - \lambda_1 (\underline{e}^T \underline{w} - 1) - \lambda_2 (\underline{\mu}^T \underline{w} - \alpha)$$

<u>FOC:</u> $L'(\underline{w}) =$

<u>C:</u>

$$\boxed{\begin{aligned} 2\Sigma \cdot \underline{w} - \lambda_1 \underline{e} - \lambda_2 \cdot \mu &= \underline{0} \\ \underline{e}^T \cdot \underline{w} &= 1 \\ \underline{\mu}^T \cdot \underline{w} &= \alpha \end{aligned}}$$

<u>FOC:</u>

$$2\Sigma \cdot \underline{w} = \lambda_1 \underline{e} + \lambda_2 \underline{\mu} \qquad 1:2$$

$$\Sigma \cdot \underline{w} = \frac{\lambda_1}{2} \underline{e} + \frac{\lambda_2}{2} \underline{\mu}$$

$$\cancel{\Sigma^{-1}} \cancel{\Sigma} \underline{w} = \Sigma^{-1} \left(\frac{\lambda_1}{2} \underline{e} + \frac{\lambda_2}{2} \underline{\mu} \right)$$

$$\underline{w}^* = \frac{\lambda_1^*}{2} \Sigma^{-1} \underline{e} + \frac{\lambda_2^*}{2} \Sigma^{-1} \underline{\mu}$$

<u>C:</u> (1) $\underline{e}^T \cdot \left(\frac{\lambda_1}{2} \Sigma^{-1} \underline{e} + \frac{\lambda_2}{2} \cdot \Sigma^{-1} \underline{\mu} \right) = 1$

$$\frac{\lambda_1}{2} \underline{e}^T \Sigma^{-1} \underline{e} + \frac{\lambda_2}{2} \underline{e}^T \Sigma^{-1} \underline{\mu} = 1$$

$$\boxed{\lambda_1 \cdot (\underline{e}^T \Sigma^{-1} \underline{e}) + \lambda_2 \cdot (e^T \Sigma^{-1} \underline{\mu}) = 2}$$

(2) $\underline{\mu}^T \left(\frac{\lambda_1}{2} \Sigma^{-1} \underline{e} + \frac{\lambda_2}{2} \Sigma^{-1} \underline{\mu} \right) = \alpha$

$$\boxed{\lambda_1 (\underline{\mu}^T \Sigma^{-1} \underline{e}) + \lambda_2 (\underline{\mu}^T \Sigma^{-1} \mu) = 2\alpha}$$

$$\begin{pmatrix} \underline{e}^T \Sigma^{-1} \underline{e} & \underline{e}^T \Sigma^{-1} \underline{\mu} \\ \underline{\mu}^T \Sigma^{-1} \underline{e} & \underline{\mu}^T \Sigma^{-1} \underline{\mu} \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2\alpha \end{pmatrix}$$

It follows from the assumptions that

$$\begin{pmatrix} \underline{e}^T \Sigma^{-1} \underline{e} & \underline{e}^T \Sigma^{-1} \underline{\mu} \\ \underline{\mu}^T \Sigma^{-1} \underline{e} & \underline{\mu}^T \Sigma^{-1} \underline{\mu} \end{pmatrix} = \underbrace{\begin{pmatrix} \underline{e}^T \\ \underline{\mu}^T \end{pmatrix} \Sigma^{-1} \cdot (\underline{e} \mid \underline{\mu})}_{\text{pos. defn.}}$$

is invertible, hence

$$\begin{pmatrix} \lambda_1^* \\ \lambda_2^* \end{pmatrix} = \begin{pmatrix} \underline{e}^T \Sigma^{-1} \underline{e} & \underline{e}^T \Sigma^{-1} \underline{\mu} \\ \underline{\mu}^T \Sigma^{-1} \underline{e} & \underline{\mu}^T \Sigma^{-1} \underline{\mu} \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 \\ 2\alpha \end{pmatrix}$$

Hence there is one candidate pt $(\underline{w}^*; \lambda_1^*, \lambda_2^*)$.

<u>SOC:</u> $h = \underline{w}^T \Sigma \underline{w} - \underbrace{\lambda_1^* (\underline{e}^T \underline{w} - 1) - \lambda_2^* (\underline{\mu}^T \underline{w} - \alpha)}_{\text{linear in } \underline{w}}$

$H(h) = 2\Sigma$ pos. defn.

$\Downarrow$

h convex

$\Downarrow$ SOC

$(\underline{w}^*; \lambda_1^*, \lambda_2^*)$ is min. pt.

<u>Recall:</u> $\underline{\mu}, \Sigma$ given s.t. the following assumptions hold:

i) Σ symmetric and pos. semidefn.

ii) Σ pos. defn.

iii) $\underline{\mu}, \underline{e}$ linearly independent