Key Problems

Problem 1.
Write Python code that defines a function `position(vector,n)`, where `vector` is a one-dimensional ndarray, assumed to be sorted in ascending order, and where `n` is an integer. The function should return the minimal integer i such that n <= `vector`[i], or the length of `vector` is no such integer i exists. Use recursion. Check that `position(u,n)` returns 0 if n = 2, 3 if n = 6, and 6 if n = 21 when `u = np.array([2,3,5,6,12,17])`.

Problem 2.
Write Python code that defines a function `insert(vector,n)`, where `vector` is a one-dimensional ndarray, assumed to be sorted in ascending order, and where `n` is an integer. The function should return a one-dimensional ndarray, sorted in ascending order, containing all elements from `vector` and `n`. Hint: use `position` and `np.append`. Check that `insert(u,n)` returns what you expect when `u = np.array([2,3,5,6,12,17])` and n = 1, n = 6 and n = 21.

Problem 3.
Write Python code that defines a function `Gauss(matrix)` which returns an echelon form of the matrix `matrix`. Use recursion. Then use this code to find an echelon form of the following matrices:

a) `A = np.array([[1,1,1,3,-1],[1,2,4,7,3],[2,3,5,11,3]])`

b) `B = np.random.randn(10,5)`

Problem 4.
Find all complex roots of the following equations, and make a figure that shows the roots in the complex plane:

a) $x^4 + 16 = 0$ b) $x^4 + x^2 - 2 = 0$ c) $x^5 = 4 + 4i$

Problem 5.
Compute the rank of the following matrix with complex coefficients:

a) $\begin{pmatrix} 1 & i & 1 \\ i & 1 & i \\ 1 & -i & 1 \end{pmatrix}$ b) $\begin{pmatrix} i & 2 & 2 \\ 2 & i & 2 \\ 2 & 2 & i \end{pmatrix}$ c) $\begin{pmatrix} 1 & i & i & 1 \\ i & -1 & 1 & i \\ i & 1 & -1 & i \\ 1 & i & i & 1 \end{pmatrix}$

Problem 6.

a) Let ω be the complex number with polar coordinates given by $r = 1$ and $\theta = 120°$. Write $z = \omega$ and $z = \omega^2$ in the form $z = a + ib$, and draw a figure of the complex plane where the points $z = 1$, $z = \omega$ and $z = \omega^2$ are marked.

b) Explain that $1, \omega, \omega^2$ are the complex solutions of $x^3 = 1$. Use this to show that if $z = z^*$ is one solution of the equation $x^3 = -i$, then the other solutions of this equation are $z = z^* \cdot \omega$ and $z = z^* \cdot \omega^2$.

c) Compute $(2 + i)^3$, and use this computation to find all complex solutions of $x^3 = 2 + 11i$.

d) When Cardano's formula is applied to the equation $x^3 = 15x + 4$, it gives the following expression for solutions:

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

Some background on Cardano's formula is given in the text below. Find a way to interpret the expression above so that it makes sense, and use this to find all complex solutions of $x^3 = 15x + 4$ expressed in the form $z = a + ib$.

Cardano's formula

Let $x^3 = px + q$ be a cubic equation with real coefficients p,q. Cardano's formula for solutions is

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}$$

In fact, if we substitute $x = u + v$ in the equation, we get $u^3 + 3u^2v + 3uv^2 + v^3 = p(u + v) + q$, and this can be written as

$$(u^3 + v^3 - q) + (u + v)(3uv - p) = 0$$

Hence choosing u and v such that $u^3 + v^3 - q = 0$ and $3uv - p = 0$ will be sufficient for $x = u + v$ to be a solution. The second equation gives $v = p/(3u)$, and we substitute this in the first equation:

$$u^3 + \left(\frac{p}{3u}\right)^3 - q = 0 \quad \Rightarrow \quad u^6 - qu^3 + \left(\frac{p}{3}\right)^3 = 0$$

This is a quadratic equation in u^3, and we find that

$$u^3 = \frac{q \pm \sqrt{(-q)^2 - 4(p/3)^3}}{2} = \frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}$$

Since $u^3 + v^3 = q$, we may without loss of generality assume that

$$u^3 = \frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}, \quad v^3 = \frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}$$

Since $x = u + v$, we add the third roots of the expressions above, and this gives Cardano's formula:

$$x = u + v = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}$$

Exercise problems

Problems from the textbook: [E] A.1 - A.6

Answers to Key Problems

Problem 4.

a) $\sqrt{2}(1 + i), \sqrt{2}(1 - i), -\sqrt{2}(1 + i), -\sqrt{2}(i - 1)$

b) $1, -1, \sqrt{2}i, -\sqrt{2}i$

c) $\sqrt{2}(\cos(\theta_i) + i\sin(\theta_i))$ for $i = 0$,1,2,3,4, with $\theta_i = 9° + i \cdot 72°$

Problem 5.

a) 2

b) 3

c) 3

Problem 6.

$x = 4,\ x = -2 + \sqrt{3},\ x = -2 - \sqrt{3}$