Optimal portfolio theory

We are to construct a portfolio, and assume that there are n assets to choose from. We assume that asset i has return R_i, which is a stochastic variable with

$$E(R_i) = \mu_i \quad , \quad Var(R_i) = \sigma_i^2$$

for constant and known parameters μ_i, σ_i. We also assume that the covariance

$$Cov(R_i, R_j) = \sigma_{ij}$$

is constant and known.

We choose portfolio weights $w_1, w_2, \ldots, w_n$ with $w_1 + w_2 + \ldots + w_n = 1$, where w_i is the share of our capital that are used to buy asset i. We allow $w_i < 0$ (short selling). For example

$$w_1 = w_2 = \ldots = w_n = 1/n$$

means that we use the same capital in each of the i assets.

Example I: 3 assets, monthly returns

Asset	μ_i	σ_i
1	0.0427	0.1000
2	0.0015	0.1044
3	0.0285	0.1411

$\sigma_{12} = 0.0018$

$\sigma_{13} = 0.0011$

$\sigma_{23} = 0.0026$

Some important formulas:

Let R be the return of the portfolio. Then $R = \omega_1 R_1 + \omega_2 R_2 + \dots + \omega_n R_n$.

We have:

i) $$E(R) = \omega_1 E(R_1) + \omega_2 E(R_2) + \dots + \omega_n E(R_n)$$
$$= \omega_1 \mu_1 + \omega_2 \mu_2 + \dots + \omega_n \cdot \mu_n$$
$$= (\mu_1\ \mu_2\ \dots\ \mu_n) \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix} = \underline{\mu}^T \cdot \underline{\omega}$$

ii) $$Var(R) = \omega_1^2 Var(R_1) + \omega_1 \omega_2 Cov(R_1, R_2) + \dots + \omega_n^2 Var(R_n)$$
$$= \omega_1^2 \sigma_1^2 + \omega_1 \omega_2 \sigma_{12} + \dots + \omega_n^2 \sigma_n^2$$
$$= (\omega_1 . \omega_2 \dots \omega_n) \cdot \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \dots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{2n} & \dots & \sigma_n^2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix}$$
$$= \underline{\omega}^T \cdot \Sigma\ \underline{\omega}$$

We have used the matrices:

$$\underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix}$$

The covariance-matrix Σ is a positive semidefinite symmetric matrix in general, since

$$Var(R) = \underline{w}^T \Sigma \underline{w} \geq 0 \quad \text{for all } \underline{w}$$

In example I, we have:

$$\underline{\mu} = \begin{pmatrix} 0.0427 \\ 0.0015 \\ 0.0285 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 0.1000^2 & 0.0018 & 0.0011 \\ 0.0018 & 0.1044^2 & 0.0026 \\ 0.0011 & 0.0026 & 0.1411^2 \end{pmatrix}$$

$\boxed{\begin{array}{l} \underline{\text{Additional assumptions:}} \\ \text{In what follows, we use the additional assumptions:} \\ \text{i) } \Sigma \text{ positive definite} \\ \text{ii) } \left\{ \underline{\mu}, \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\} \text{ are linearly independent} \end{array}}$

Remark: i) means $|\Sigma| \neq 0$

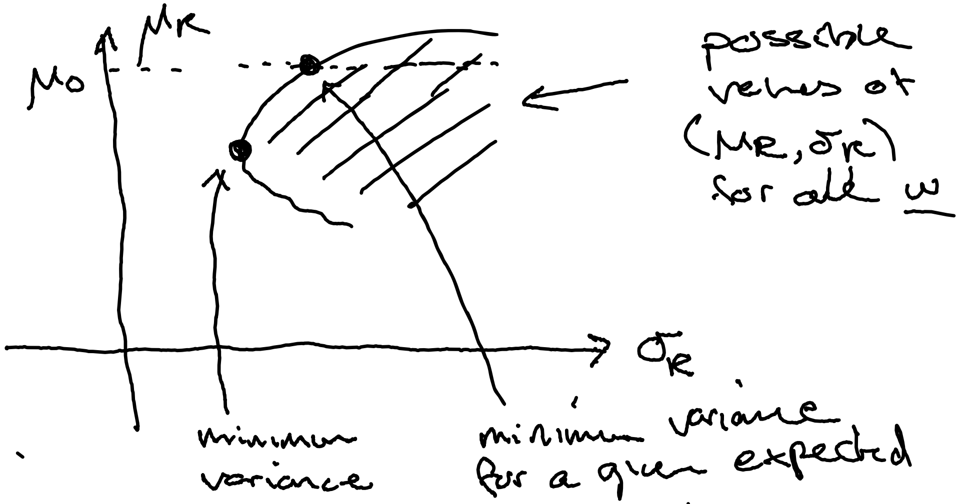
ii) means $\mu \neq c \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

We think of the expected return and variance of the portfolio as functions of $\underline{w}$:

$$\mu(R) = \underline{\mu}^T \cdot \underline{w}$$
$$Var(R) = \underline{w}^T \Sigma \underline{w}$$

and write $\mu_R = \underline{\mu}^T \cdot \underline{w}$, $\sigma_R = \sqrt{\underline{w}^T \Sigma \underline{w}}$

For each $\underline{w}$ (chosen portfolio), we can plot expected return / std deviation in this coordinate system:



Problems:

a) $\min \underline{w}^T \Sigma \underline{w}$ when $w_1 + w_2 + \ldots + w_n = 1$

b) $\min \underline{w}^T \Sigma \underline{w}$ when $\begin{cases} w_1 + \ldots + w_n = 1 \\ \underline{\mu}^T \cdot \underline{w} = \mu_0 \end{cases}$

Solution of a):

Let $\underline{u} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ be a vectors of 1's.

Then we can write the problem:

$$\boxed{\min \underline{\omega}^T \Sigma \underline{\omega} \quad \text{wh} \quad \underline{u}^T \cdot \underline{\omega} = 1}$$

This is a Lagrange problem with

$$L = \underline{\omega}^T \Sigma \underline{\omega} - \lambda \cdot (\underline{u}^T \cdot \underline{\omega})$$

see note end of next page

and we have:

FOC: $L'_{\underline{\omega}} = \boxed{2 \Sigma \underline{\omega} - \lambda \underline{u} = \underline{0}}$

c: $\boxed{\underline{u}^T \cdot \underline{\omega} = 1}$

FOC's give: $\Sigma \underline{\omega} = \frac{\lambda}{2} \underline{u}$

use that $|\Sigma| \neq 0$ $\longrightarrow$ $\underline{\omega} = \Sigma^{-1}\left(\frac{\lambda}{2} \underline{u}\right) = \frac{\lambda}{2} \Sigma^{-1} \cdot \underline{u}$

c: $\underline{u}^T \cdot \underline{\omega} = \underline{u}^T \left(\frac{\lambda}{2} \Sigma^{-1} \cdot \underline{u}\right) = 1$

$$\frac{\lambda}{2} \underline{u}^T \cdot \Sigma^{-1} \cdot \underline{u} = 1$$

$$\frac{\lambda}{2} = \frac{1}{\underline{u}^T \Sigma^{-1} \cdot \underline{u}}$$

$$\Rightarrow \underline{\omega} = \frac{1}{\underline{u}^T \Sigma^{-1} \underline{u}} \cdot \Sigma^{-1} \underline{u} = \frac{\Sigma^{-1} \cdot \underline{u}}{\underline{u}^T \Sigma^{-1} \underline{u}}$$

This means that there is a unique solution of FOC+C, given by:

$$\underline{w} = \frac{\Sigma^{-1} \cdot \underline{u}}{\underline{u}^T \Sigma^{-1} \cdot \underline{u}} \qquad \text{with } \underline{u} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\lambda = \frac{2}{\underline{u}^T \Sigma^{-1} \cdot \underline{u}}$$

To prove that this is minimum, we use SOC:

$$L(\underline{w}, \lambda^*) = h(\underline{w}) = \underline{w}^T \Sigma \underline{w} - \lambda^* \underline{u}^T \cdot \underline{w}$$

$H(h) = 2\Sigma$, which is pos. defin. since Σ pos. defin.

$\Downarrow$

h convex

$\Downarrow$

$$\underline{w}^* = \frac{\Sigma^{-1} \underline{u}}{\underline{u}^T \cdot \Sigma^{-1} \cdot \underline{u}}$$ is the minimum variance portfolio

Note: We used the formulas

$f(\underline{x}) = \underline{x}^T A \underline{x}$ quadratic form $\Rightarrow \frac{\partial f}{\partial \underline{x}} = 2A \cdot \underline{x}$

$f(\underline{x}) = B \cdot \underline{x}$ linear form $\Rightarrow \frac{\partial f}{\partial \underline{x}} = B^T$.

where $\frac{\partial f}{\partial \underline{x}} = \begin{pmatrix} f'_{x_1} \\ \vdots \\ f'_{x_n} \end{pmatrix}$

Solution of b):

$$\min \underline{w}^T \Sigma \underline{w} \quad \text{when} \quad \begin{cases} \underline{u}^T \cdot \underline{w} = 1 \\ \underline{\mu}^T \cdot \underline{w} = \mu_0 \end{cases}$$

Lagrange problem:

$$L = \underline{w}^T \Sigma \underline{w} - \lambda_1 \underline{u}^T \cdot \underline{w} - \lambda_2 \cdot \underline{\mu}^T \cdot \underline{w}$$

FOC: $$L'_{\underline{w}} = 2\Sigma \cdot \underline{w} - \lambda_1 \underline{u} - \lambda_2 \cdot \underline{\mu} = \underline{0}$$

e: $$\begin{cases} \underline{u}^T \cdot \underline{w} = 1 \\ \underline{\mu}^T \cdot \underline{w} = \mu_0 \end{cases}$$

$$2\Sigma \cdot \underline{w} = \lambda_1 \underline{u} + \lambda_2 \underline{\mu}$$

$$\Sigma \underline{w} = \frac{\lambda_1}{2} \underline{u} + \frac{\lambda_2}{2} \underline{\mu}$$

$$\underline{w} = \Sigma^{-1}\left(\frac{\lambda_1}{2} \underline{u} + \frac{\lambda_2}{2} \underline{\mu}\right) = \frac{\lambda_1}{2} \Sigma^{-1} \underline{u} + \frac{\lambda_2}{2} \Sigma^{-1} \underline{\mu}$$

$\Downarrow$

$$\underline{u}^T \underline{w} = \underline{u}^T \left(\frac{\lambda_1}{2} \Sigma^{-1} \underline{u} + \frac{\lambda_2}{2} \Sigma^{-1} \underline{\mu}\right) = 1$$

$$\frac{\lambda_1}{2}\left(\underline{u}^T \Sigma^{-1} \underline{u}\right) + \frac{\lambda_2}{2}\left(\underline{u}^T \Sigma^{-1} \underline{\mu}\right) = 1$$

$$\underline{\mu}^T \underline{w} = \underline{\mu}^T \left(\frac{\lambda_1}{2} \Sigma^{-1} \underline{u} + \frac{\lambda_2}{2} \Sigma^{-1} \underline{\mu}\right) = \mu_0$$

$$\frac{\lambda_1}{2}\left(\underline{\mu}^T \Sigma^{-1} \underline{u}\right) + \frac{\lambda_2}{2}\left(\underline{\mu}^T \Sigma^{-1} \underline{\mu}\right) = \mu_0$$

We multiply these equations by 2:

$$\lambda_1 \cdot \left(\underline{u}^T \Sigma^{-1} \underline{u}\right) + \lambda_2 \left(\underline{u}^T \Sigma^{-1} \underline{\mu}\right) = 2$$

$$\lambda_1 \cdot \left(\underline{\mu}^T \Sigma^{-1} \underline{u}\right) + \lambda_2 \left(\underline{\mu}^T \Sigma^{-1} \underline{\mu}\right) = 2\mu_0$$

$$\begin{pmatrix} \underline{u}^T \Sigma^{-1} \underline{u} & \underline{u}^T \Sigma^{-1} \underline{\mu} \\ \underline{\mu}^T \Sigma^{-1} \underline{u} & \underline{\mu}^T \Sigma^{-1} \underline{\mu} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2\mu_0 \end{pmatrix}$$

$$B \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2\mu_0 \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = B^{-1} \begin{pmatrix} 2 \\ 2\mu_0 \end{pmatrix}$$

since B is positive defn $(|B| \neq 0)$:

$$B = (\underline{u} | \underline{\mu})^T \cdot \Sigma^{-1} \cdot (\underline{u} | \underline{\mu})$$

$$\Downarrow$$

$$\begin{aligned} \underline{z}^T B \underline{z} &= (z_1 \; z_2) \, B \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ &= (z_1 \; z_2) \begin{pmatrix} \underline{u}^T \\ \underline{\mu}^T \end{pmatrix} \Sigma^{-1} (\underline{u} | \underline{\mu}) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ &= (z_1 \underline{u} + z_2 \underline{\mu})^T \cdot \Sigma^{-1} \cdot (z_1 \underline{u} + z_2 \underline{\mu}) > 0 \end{aligned}$$

when $(z_1, z_2) \neq (0,0)$, since this implies

$z_1 \underline{u} + z_2 \underline{\mu} \neq (0,0)$

use iii) that $\underline{u}, \underline{\mu}$ are linearly independent + i) Σ pos. definite

and Σ pos. defn. $\Rightarrow$ Σ^{-1} pos. defn.

Conclusion:

unique solution

$$\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = B^{-1}\cdot\begin{pmatrix}2\\ 2\mu_0\end{pmatrix}$$

$\Downarrow$

unique solution for $\underline{w}$:

$$\underline{w}^* = \frac{\lambda_1}{2}\Sigma^{-1}\underline{\mu} + \frac{\lambda_2}{2}\Sigma^{-1}\underline{\mu}$$

$$= \frac{1}{2}\left(\Sigma^{-1}\underline{u} \;\; \Sigma^{-1}\underline{\mu}\right)\cdot\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}$$

$$= \underline{\frac{1}{2}\left(\Sigma^{-1}\underline{u} \;\; \Sigma^{-1}\underline{\mu}\right)\cdot B^{-1}\cdot\begin{pmatrix}2\\ 2\mu_0\end{pmatrix}}$$

As in case i), SOC gives that $\underline{w}^*$ is the <u>minimum variance portfolio</u> among portfolios with $E(R) = \mu_0$.