SYSTEMS OF LINEAR EQUATIONS
EXERCISES — FORK1003

RUNAR ILE

1. First lecture

Exercise 1.1. Which of these equations are linear equations?

(a) $2x - 11y = z$ (b) $-14x_1 + 76x_2 = 3 - x_3 - 7x_2$

(c) $\lambda x - \beta y - z = t$ (d) $\sqrt{3}x_1 + \pi x_2 - x_3 = 2x_2$

(e) $2x_2x_1 - 11x_3 = 0$ (f) $5000x_1 + 7000x_2^2 - 11000x_3 = 10^5$

Exercise 1.2. Write up the augmented matrix of each of the following systems of linear equations.

(a) $\begin{cases} 8x_1 - 11x_2 + 2x_3 - 25x_4 = 50 \\ -3x_1 + x_2 + x_3 - x_4 = -12 \end{cases}$ (b) $\begin{cases} \pi x_1 + 2x_2 + 27x_3 = 2 \\ -x_1 + 22x_2 - 3x_3 = -1 \\ 17x_2 - 15x_3 = 0 \\ \pi x_1 + x_3 = 84 \end{cases}$

(c) $\begin{cases} 5x_2 + 9x_3 = -3 \\ -4x_1 + 12x_2 - 4x_3 = -8 \end{cases}$ $\begin{matrix} x_1 + 22x_2 - 4x_3 = 5 \\ -13x_1 + 5x_3 = -8 \end{matrix}$

(d) $\begin{cases} 15x_2 + 2x_3 = -3x_1 + x_4 - 7 \\ x_2 = -3x_3 + 1 \end{cases}$ (e) $\begin{cases} 15x_2 + 2x_3 = -3x_2 + x_4 - 7 \\ x_2 = -3x_3 + 1 - x_1 \\ 0 = 2x_1 - 3x_3 + 11x_1 - 32 + 88x_3 \end{cases}$

Exercise 1.3. Write up the linear systems corresponding to the following matrices.

(a) $\begin{bmatrix} 19 & 14 & -2 \\ 0 & 4 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 5 & 0 & 0 & 0 & 41 \\ 0 & 0 & 1 & -3 & 0 & 99 \\ 0 & 0 & 0 & 0 & 1 & 19 \end{bmatrix}$

(c) $\begin{bmatrix} 87 & -1 & 0 & 15 & 9 \\ 0 & 0 & 0 & 8 & 7.5 \\ 0 & 0 & 9 & -1 & 33 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 9 & -56 & 23 \end{bmatrix}$

(e) $\begin{bmatrix} 55 & 300 & 0 & 350 & 4000 \\ 0 & 0 & 8 & 125 & 400 \end{bmatrix}$ (f) $\begin{bmatrix} -40 & 50 & -20 & 200 & 300 \\ 0 & 30 & 0 & 90 & 110 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

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Exercise 1.4. For each matrix in Exercise 1.3 determine if it's in echelon form or in reduced echelon form.

Exercise 1.5. For each of the following matrices on reduced echelon form determine the pivot columns and the free variables. Use this to solve the corresponding system of linear equations.

(a) $\begin{bmatrix} 1 & 0 & -2 & 7 \\ 0 & 1 & 8 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 0 & 0 & 5 & 36 \\ 0 & 0 & 1 & 0 & 2 & 29 \\ 0 & 0 & 0 & 1 & 1 & 19 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -25 & 0 & 3.5 & 400 \\ 0 & 0 & 1 & -40 & 300 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 & 0 & 37 \\ 0 & 1 & 10 & 0 & 400 \\ 0 & 0 & 0 & 1 & \sqrt{2} \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 1 & 0 & -5 & 14 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Exercise 1.6. Hege and Kåre are solving systems of linear equations by

(1) writing up the augmented matrix to the linear system on standard form,
(2) doing Gauss elimination (row reduction) on the augmented matrix as to obtain a matrix on reduced echelon form,
(3) solving the corresponding equivalent linear system.

Hege is doing elementary row operations. Explain in each case which operation she has performed and determine whether this brings the Gauss elimination forward or not.

(a) $\begin{bmatrix} 1 & 5 & -9 & 3 & 6 \\ 0 & 2 & 7 & -4 & -12 \\ -5 & -4 & -1 & 3 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -9 & 3 & 6 \\ 0 & 2 & 7 & -4 & -12 \\ 0 & 21 & -46 & 18 & 40 \end{bmatrix}$

(b) $\begin{bmatrix} 7 & 35 & -14 & 70 & -21 \\ 0 & 0 & 6 & -4 & 16 \\ 0 & 1 & -1 & 5 & 0 \\ 1 & 5 & -2 & 10 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -2 & 10 & -3 \\ 0 & 0 & 6 & -4 & 16 \\ 0 & 1 & -1 & 5 & 0 \\ 7 & 35 & -14 & 70 & -21 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -12 & -6 & 0 & -33 \\ 0 & 7 & -5 & 2 & 24 \\ 2 & -8 & -1 & 5 & -22 \end{bmatrix} \sim \begin{bmatrix} 3 & -12 & -6 & 0 & -33 \\ 0 & 7 & -5 & 2 & 24 \\ 0 & 0 & 1 & 5 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 0 & 3 & -15 \\ 0 & 0 & -2 & 14 & -6 \\ 0 & 5 & 7 & 8 & -30 \\ 8 & -3 & 0 & 2 & 72 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 3 & -15 \\ 0 & 5 & 7 & 8 & -30 \\ 0 & 0 & -2 & 14 & -6 \\ 8 & -3 & 0 & 2 & 72 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 7 & 5 & -2 & -11 \\ 6 & 9 & 15 & 27 & -18 \\ -4 & -6 & 11 & 16 & -19 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 5 & -2 & -11 \\ 6 & 9 & 15 & 27 & -18 \\ 0 & 0 & 21 & 34 & -25 \end{bmatrix}$

(f) $\begin{bmatrix} 3 & 7 & 0 & 9 & 17 \\ 0 & 14 & 0 & 2 & 4 \\ 0 & 0 & 1 & 0 & 19 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 0 & 8 & 15 \\ 0 & 14 & 0 & 2 & 4 \\ 0 & 0 & 1 & 0 & 19 \end{bmatrix}$

(g) $\begin{bmatrix} 5 & 3 & 6 & 7 & 18 & 23 \\ 0 & 11 & 0 & 2 & -1 & 6 \\ 0 & 0 & -6 & 3 & 24 & 15 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 5 & 3 & 0 & 10 & 42 & 38 \\ 0 & 11 & 0 & 2 & -1 & 6 \\ 0 & 0 & -6 & 3 & 24 & 15 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$

(h) $\begin{bmatrix} 80 & 0 & 0 & 120 & 80 & 60 \\ 0 & 45 & 0 & 15 & 20 & 35 \\ 0 & 0 & 50 & 30 & 20 & 15 \end{bmatrix} \sim \begin{bmatrix} 80 & 0 & -200 & 0 & 0 & 0 \\ 0 & 45 & 0 & 15 & 20 & 35 \\ 0 & 0 & 50 & 30 & 20 & 15 \end{bmatrix}$

(i) $\begin{bmatrix} -3 & 6 & 0 & 9 & 3 & 15 \\ 0 & 0 & 1 & 9 & 3 & 15 \end{bmatrix} \sim \begin{bmatrix} -3 & 6 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 9 & 3 & 15 \end{bmatrix}$

(j) $\begin{bmatrix} 0 & 3 & 7 & 2 & 0 & 27 \\ 5 & 0 & 7 & 2 & 4 & 27 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 & 7 & 2 & 4 & 27 \\ 0 & 3 & 7 & 2 & 0 & 27 \end{bmatrix}$

Kåre is not always taking sufficiently care with his elementary row operations. In each of the following transformations explain what Kåre does, determine whether the transformation is an elementary row operation, and determine whether the matrices are row equivalent.

(k) $\begin{bmatrix} 4 & 3 & 1 & 11 \\ 0 & 5 & 9 & -2 \\ -12 & 5 & -7 & 30 \end{bmatrix} \sim \begin{bmatrix} 4 & 3 & 1 & 11 \\ 0 & 5 & 9 & -2 \\ 0 & -4 & -10 & -3 \end{bmatrix}$

(l) $\begin{bmatrix} 5 & 3 & 0 & 10 \\ -15 & 8 & 7 & -4 \\ 10 & 0 & -3 & 20 \end{bmatrix} \sim \begin{bmatrix} 5 & 3 & 0 & 10 \\ 0 & 17 & 7 & 26 \\ 0 & -6 & -3 & 0 \end{bmatrix}$

(m) $\begin{bmatrix} 3 & 7 & -6 & 4 & 3 \\ 0 & 0 & 7 & -4 & 5 \\ 3 & 7 & 1 & 0 & 8 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & -7 & 4 & -5 \\ 0 & 0 & 7 & -4 & 5 \\ 0 & 0 & 7 & -4 & 5 \end{bmatrix}$

(n) $\begin{bmatrix} 3 & 0 & -3 \\ 0 & 2 & 2 \\ 2 & 3 & 1 \\ 7 & 16 & 9 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \\ 2 & 3 & 3 \\ 7 & 16 & 16 \end{bmatrix}$

(o) $\begin{bmatrix} 8 & 1 & -2 & 5 & 1 \\ 9 & -9 & 4 & -4 & 2 \\ 8 & 1 & 7 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 9 & -9 & 4 & -4 & 2 \\ 0 & 0 & 9 & -3 & 6 \end{bmatrix}$

Exercise 1.7. Use Gauss elimination to find row equivalent matrices on reduced echelon form.

(a) $\begin{bmatrix} 1 & -2 & 0 & -2 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 4 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 & 0 & -2 \\ 1 & -1 & -3 & -5 \\ -1 & 0 & 10 & 16 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 4 & 26 & 0 & 56 \\ 0 & 1 & 5 & -2 & -9 \\ 0 & 2 & 10 & -1 & 12 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 16 & 1 & 44 \\ 2 & 9 & 57 & -2 & 103 \\ 2 & 8 & 52 & 0 & 112 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 0 & 0 & 7 & 21 \\ 0 & 0 & 3 & -6 & -6 \\ 1 & 2 & -3 & 3 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} 4 & 8 & -6 & 7 & 17 \\ 1 & 2 & 0 & -3 & -4 \\ 3 & 6 & -9 & 9 & 6 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 1 & 2 & 23 \\ 0 & 1 & 2 & 16 \\ 5 & 0 & 1 & 40 \end{bmatrix}$ (h) $\begin{bmatrix} 5 & 5 & 10 & 115 \\ 5 & 10 & 20 & 195 \\ 5 & 1 & 3 & 56 \end{bmatrix}$

Exercise 1.8. The norwegian economist Trygve M. Haavelmo (Nobel laureate 1989) devised a model for the U.S. economy for the years 1929-1941 based on the following equations:

(i) $c = 0.712y + 95.05$ (ii) $s = 0.158(c + x) - 34.30$

(iii) $y = c + x - s$ (iv) $x = 93.53$

Here x denotes total investment, y is disposable income, s is the total savings by firms, and c is total consumption. Write the system of linear equations in standard form with the variables in the order x, y, s, and c. Solve the system.[1]

[1]From Sydsæter and Hammond, *Essential Mathematics for Economics*, Chap. 15