FORK1003
LECTURE DAY 1 – OVERVIEW

RUNAR ILE

1. Systems of linear equations

1.1. What is a linear equation?

Example 1.1. Variables and constraint.

$$3x + y - 2z = 15$$

Example 1.2. The variables of degree 1 only and only multiplied with numbers. These are *not* linear equations:

$$3x + y - 2z^3 = 15 \qquad xy + 30z = 9 \qquad 2000x - 380y = 10^z$$

Example 1.3. Notation for many variables.

$$3x_1 + x_2 - 2x_3 = 15 \qquad 37x_1 - 25x_2 + 100x_3 - 2x_4 + x_5 - 93x_6 = 0$$

Exercise 1.4. Is this a linear equation?

$$\sqrt{5}x_1 - 11x_3 + 23 = -2x_3 + 21.9 + 10^8 x_4 + \frac{2x_2}{7}$$

Example 1.5. Parametres are like numbers, not variables.

$$\alpha x_1 - \beta x_2 + 10000x_3 = \gamma$$

Example 1.6. The general linear equation.

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b \qquad \text{or} \qquad \sum_{i=1}^{i=n} a_ix_i = b$$

1.2. What is a system of linear equations?

Example 1.7. Variables and several constraints.

$$\left\{\begin{array}{llll} 3x_1 & +x_2 & -2x_3 & = 15 \\ 2x_1 & -x_2 & -11x_3 & = 14 \\ -20x_1 & +3x_2 & +5x_3 & = 9 \end{array}\right.$$

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1.3. **To solve a system of linear equations.** What does it mean to *solve* a system of linear equations?

Definition 1.8. Two systems (of linear equations) are *equivalent* if they have the same solutions.

Example 1.9. To solve a system by finding simpler, equivalent systems.

$$\text{Which numbers } x_1 \text{ and } x_2 \text{ satisfy } \begin{cases} 2x_1 & -3x_2 & = 6 \\ 4x_1 & +9x_2 & = 72 \end{cases} \quad ?$$

$$\text{An equivalent system: } \begin{cases} 4x_1 & -6x_2 & = 12 \\ 4x_1 & +9x_2 & = 72 \end{cases}$$

$$\text{An equivalent system: } \begin{cases} 4x_1 & -6x_2 & = 12 \\ & 15x_2 & = 60 \end{cases}$$

$$\text{An equivalent system: } \begin{cases} 4x_1 & -6x_2 & = 12 \\ & x_2 & = 4 \end{cases}$$

$$\text{An equivalent system: } \begin{cases} 4x_1 & & = 36 \\ & x_2 & = 4 \end{cases}$$

$$\text{An equivalent system: } \begin{cases} x_1 & & = 9 \\ & x_2 & = 4 \end{cases}$$

$$\text{We check with the original system: } \begin{cases} 2\cdot 9 & -3\cdot 4 & = 6 \\ 4\cdot 9 & +9\cdot 4 & = 72 \end{cases} \quad \text{Ok!}$$

There are several ways to do this reduction. One can also consider the geometry. Each equation has a line of solutions. The intersection in the plane of the two lines of solutions gives the solution of the system. Draw the lines in a coordinate system!

To find an equivalent system there are two basic things to do:

- Multiply an equation with a *non-zero* number.
- Add (or subtract) a multiple of an equation to *another* equation.

1.4. **From systems to matrices of numbers.** If the system is on standard form, we don't really need to write the variables or =.

Example 1.10.

$$\begin{cases} 3x_1 & +x_2 & -2x_3 & = 15 \\ 2x_1 & -x_2 & -11x_3 & = 14 \\ -20x_1 & +3x_2 & +5x_3 & = 9 \end{cases} \quad \rightsquigarrow \quad \begin{bmatrix} 3 & 1 & -2 & 15 \\ 2 & -1 & -11 & 14 \\ -20 & 3 & 5 & 9 \end{bmatrix}$$

This is the *augmented matrix* of the system.

Example 1.11. To find equivalent systems by finding equivalent matrices.

$$\begin{cases} 2x_1 & -3x_2 & = 6 \\ 4x_1 & +9x_2 & = 72 \end{cases} \qquad \begin{bmatrix} 2 & -3 & 6 \\ 4 & 9 & 72 \end{bmatrix}$$

$$\begin{cases} 4x_1 & -6x_2 & = 12 \\ 4x_1 & +9x_2 & = 72 \end{cases} \qquad \begin{bmatrix} 4 & -6 & 12 \\ 4 & 9 & 72 \end{bmatrix}$$

$$\begin{cases} 4x_1 & -6x_2 & = 12 \\ & 15x_2 & = 60 \end{cases} \qquad \begin{bmatrix} 4 & -6 & 12 \\ 0 & 15 & 60 \end{bmatrix}$$

$$\begin{cases} 4x_1 & -6x_2 & = 12 \\ & x_2 & = 4 \end{cases} \qquad \begin{bmatrix} 4 & -6 & 12 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{cases} 4x_1 & & = 36 \\ & x_2 & = 4 \end{cases} \qquad \begin{bmatrix} 4 & 0 & 36 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{cases} x_1 & & = 9 \\ & x_2 & = 4 \end{cases} \qquad \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & 4 \end{bmatrix}$$

What you can do with systems, you can also do with augmented matrices.

To find a new matrix with equivalent associated system there are three basic things to do (corresponding to operations which produce equivalent systems):

- Multiply (or divide) a row with a *non-zero* number.
- Add (or subtract) a multiple of a row to *another* equation.
- Interchange two rows.

These are called the *elementary row operations.* Note that the undoing of an elementary row operation also is an elementary row operation.

Definition 1.12. Two matrices **A** and **B** are *row equivalent* if **B** can be obtained from **A** by a sequence of elementary row operations.

Fact 1: Row equivalent matrices have equivalent associated systems.
Fact 2: Equivalent systems have row equivalent augmented matrices.

1.5. **Matrices can have (reduced) echelon form.** For some matrices it's easier to solve the system of linear equations.

- The *pivot* (leading entry) is the first non-zero entry from the left in a row.

Definition 1.13. Consider the following properties of a matrix.

(1) The zero rows (all entries are zero) are below the non-zero rows.
(2) The pivot of a row is strictly to the right of the pivots in the rows above.
(3) The pivots are all equal to 1.
(4) The entries in the column above any pivot are zero.

A matrix has *row echelon form* if (1)-(2) hold. A matrix has *reduced* row echelon form if (1)-(4) hold.

Easy characteristic of row echelon form: The *staircase* of the pivot positions should only have gentle steps.

Example 1.14. See Echelon1.

1.6. **Gauss-Jordan elimination of matrices.** A systematic way to solve all systems:
Step 1: Write up the augmented matrix of the system.
Use elementary row operations to:
  Step 2: Find a row equivalent matrix on row echelon form.
  Step 3: Find a row equivalent matrix on reduced row echelon form.
Step 4: Solve the associated system of the matrix obtained in Step 3.

Example 1.15. See Gauss1.

Example 1.16. Note that there can be *no* solutions to a system:

$$\begin{bmatrix} 3 & 7 & 12 & 21 \\ 2 & 6 & 8 & 13 \\ 1 & 1 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 3 & 7 & 12 & 21 \\ 2 & 6 & 8 & 13 \\ 3 & 7 & 12 & 20 \end{bmatrix} \sim \begin{bmatrix} 3 & 7 & 12 & 21 \\ 2 & 6 & 8 & 13 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The corresponding system $\begin{cases} 3x_1 & +7x_2 & 12x_3 & = 21 \\ 2x_1 & +6x_2 & +8x_3 & = 13 \\ 0x_1 & +0x_2 & +0x_3 & = 1 \end{cases}$

has no solutions since in the last equation the left hand side is zero while the right hand side is one. Then the original system cannot have any solutions either. (Are you sure you understand why?)

1.7. **Very good news on Gauss-Jordan elimination.** Whatever way you do your Gauss-Jordan elimination the reduced echelon form will give the same matrix (as long as you don't make mistakes):

Theorem 1.17. *A matrix is row equivalent to a* <u>*unique*</u> *matrix on reduced echelon form.*