FORK1003
LECTURE DAY 2 AND 3 – OVERVIEW

RUNAR ILE

0.1. **Systems with infinitely many solutions.** If there are more unknowns than equations *and* there is a solution of the system, then there are infinitely many solutions to the system. But we may also have infinitely many solutions in other cases:

Example 0.1. Consider the system of equations

$$\begin{cases} 4x_1 & & -12x_3 & = 40 \\ & 3x_2 & -6x_3 & = 15 \\ 2x_1 & +2x_2 & -10x_3 & = 30 \end{cases}$$

We solve the system by performing Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix} 4 & 0 & -12 & 40 \\ 0 & 3 & -6 & 15 \\ 2 & 2 & -10 & 30 \end{bmatrix} \quad \text{Divide the first row with 4}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 10 \\ 0 & 3 & -6 & 15 \\ 2 & 2 & -10 & 30 \end{bmatrix} \quad \text{Subtract 2 times the first row from the third}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 10 \\ 0 & 3 & -6 & 15 \\ 0 & 2 & -4 & 10 \end{bmatrix} \quad \text{Divide the second row by 3}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 10 \\ 0 & 1 & -2 & 5 \\ 0 & 2 & -4 & 10 \end{bmatrix} \quad \text{Subtract 2 times the second row from the third}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 10 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{This matrix is on reduced echelon form}$$

The associated system is

$$\begin{cases} x_1 & & -3x_2 & = 10 \\ & x_2 & -2x_3 & = 5 \\ 0x_1 & +0x_2 & +0x_3 & = 0 \end{cases}$$

The third equation is true for all values of the variables. The second equation can be rewritten as $x_2 = 2x_3 + 5$ and first equation as $x_1 =$

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$3x_3 + 10$. In this way x_3 is a *free* variable (can have any value), and x_1 and x_2 are determined by these equations and the value of x_3. We write:

$$\begin{cases} x_1 = 3x_3 + 10 \\ x_2 = 2x_3 + 5 \\ x_3 \text{ is a free variable} \end{cases}$$

A *pivot column* is a column with a pivot. Note that in the example the free variable is the *third* variable and this corresponds to the third column in the reduced echelon form being *not* a pivot column. The number of free variables in the solutions of a system is called *the degree of freedom* in the solutions. In the example the degree of freedom is 1.

Example 0.2. See Gauss2.

1. Matrix arithmetic

Addition is easy. Two $(m \times n)$-matrices are added by adding corresponding entries. The result is an $(m \times n)$ matrix. One can also multiply a matrix with a number by multiplying all entries with the number.

Exercise 1.1. Consider the two matrices

$$\mathbf{A} = \begin{bmatrix} 4 & -3 & 1 \\ 11 & 4 & 6 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 3 & 0 & 7 \\ 2 & 5 & -1 \end{bmatrix}.$$

Find $3\mathbf{A}$, $-\mathbf{B}$, $\mathbf{A} - 2\mathbf{B}$, $2\mathbf{A} + 3\mathbf{B}$.

Multiplication is harder.

Example 1.2. The second matrix is a column:

$$\begin{bmatrix} 6 & 7 & 8 & 2 \\ 9 & 11 & -4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 9 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \cdot 2 + 7 \cdot 3 + 8 \cdot 9 - 2 \cdot 5 \\ 9 \cdot 2 + 11 \cdot 3 - 4 \cdot 9 - 3 \cdot 5 \end{bmatrix} = \begin{bmatrix} 95 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 7 & 8 & 2 \\ 9 & 11 & -4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \cdot 3 - 7 \cdot 1 + 8 \cdot 2 + 2 \cdot 4 \\ 9 \cdot 3 - 11 \cdot 1 - 4 \cdot 2 + 3 \cdot 4 \end{bmatrix} = \begin{bmatrix} 35 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 7 & 8 & 2 \\ 9 & 11 & -4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \cdot 10 + 7 \cdot 4 + 8 \cdot 0 + 2 \cdot 3 \\ 9 \cdot 10 + 11 \cdot 4 - 4 \cdot 0 + 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 94 \\ 143 \end{bmatrix}$$

Example 1.3. The first matrix is a (2×4), the second matrix is a (4×3), the result is a (2×3). Take each column in the second matrix and multiply as in the previous example:

$$\begin{bmatrix} 6 & 7 & 8 & 2 \\ 9 & 11 & -4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 10 \\ 3 & -1 & 4 \\ 9 & 2 & 0 \\ -5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 95 & 35 & 94 \\ 0 & 20 & 143 \end{bmatrix}$$

The same method is applied to all matrix multiplication. But the two matrices have to fit together! The columns in the second matrix have to be as high as the rows of the first matrix are wide. So if the first matrix is a $(m \times n)$, the second has to be a $(n \times k)$. The result of the multiplication is then a $(m \times k)$-matrix.

Exercise 1.4. Consider the two matrices

$$\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}.$$

Find **AB**, **BA**, $\mathbf{B}^2$, $\mathbf{A}^2$, **ABB**.

Exercise 1.5. Consider the two matrices

$$\mathbf{A} = \begin{bmatrix} 2 & -5 & 1 & 3 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 6 \\ -1 \\ -2 \\ 3 \end{bmatrix}.$$

Find **AB** and **BA**.

1.1. **Systems of linear equations as matrix multiplication.** For the lazy.

Example 1.6. We write the following system by matrix multiplication.

$$\begin{cases} 3x_1 & +x_2 & -2x_3 & +5x_4 & = 15 \\ 2x_1 & -x_2 & -11x_3 & +x_4 & = 14 \\ -20x_1 & +3x_2 & +5x_3 & +7x_4 & = 9 \end{cases}$$

$$\begin{bmatrix} 3 & 1 & -2 & 5 \\ 2 & -1 & -11 & 1 \\ -20 & 3 & 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ 9 \end{bmatrix}$$

We can introduce symbols for the three matrices. Let **A** denote the (3×4)-matrix, let **x** denote the column of variables and **b** the matrix to the right. Then the equation is

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

which is the same as the system of linear equations we started with. So many linear equations become one matrix equation. The matrix **A** is called the *coefficient matrix* of the system. Note that if we extend **A** by adjoining the column $\mathbf{b}$ to the right we get the augmented matrix of the system.

2. Determinants

2.1. **The wonderful determinant.** To any *square* matrix $\mathbf{A}$ there is a number, called the *determinant* of $\mathbf{A}$, which is denoted $\det(\mathbf{A})$. It has many wonderful properties:

(1) If $\det(\mathbf{A}) \neq 0$ then the matrix equation (system of linear equations)
$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$
has a unique solution for any column vector $\mathbf{b}$.

(2) If $\mathbf{A}$ and $\mathbf{B}$ are two square matrices of the same size, then the determinant of the product equals the product of the determinants:
$$\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$$

(3) If the square matrix $\mathbf{B}$ is obtained from the square matrix $\mathbf{A}$ by multiplying a row in $\mathbf{A}$ by a number c then
$$\det(\mathbf{B}) = c \cdot \det(\mathbf{A})$$

(4) If the square matrix $\mathbf{B}$ is obtained from the square matrix $\mathbf{A}$ by adding (or subtracting) a multiple of a row to *another* row, then
$$\det(\mathbf{B}) = \det(\mathbf{A})$$

(5) If the square matrix $\mathbf{B}$ is obtained from the square matrix $\mathbf{A}$ by interchanging two rows then
$$\det(\mathbf{B}) = -\det(\mathbf{A})$$

2.2. **Can we calculate it?** Given a square matrix $\mathbf{A}$, how do we find the determinant $\det(\mathbf{A})$? One method reduces the calculation to smaller and smaller matrices. The determinant of the 1×1-matrix $[a]$ equals the a itself. The second smallest is the (2×2)-matrix:

Example 2.1. If $\mathbf{A}$ is 2×2 then the determinant is the product of the two elements on the diagonal minus the product of the other two elements:
$$\text{If } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } \det(\mathbf{A}) = ad - bc$$

Exercise 2.2. Find the determinant.
$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 6 & 3 \\ 7 & 4 \end{bmatrix}$$

Example 2.3. If $\mathbf{A}$ is 3×3 then the determinant can be calculated by the determinants of three (2×2) matrices:

$$\text{If } \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then}$$

$$\det(\mathbf{A}) = a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Can you see the pattern?

Exercise 2.4. Find the determinant.

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 0 \\ 17 & 71 & 2 \\ 0 & 19 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 0 \\ 6 & 0 & 3 \\ 0 & 5 & -4 \end{bmatrix}$$

Example 2.5. If $\mathbf{A}$ is (4×4) then the determinant can be calculated by the determinants of four (3×3) matrices:

$$\text{If } \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \text{ then}$$

$$\det(\mathbf{A}) = a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{bmatrix}$$

$$+ a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{bmatrix} - a_{14} \cdot \det \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

Do you see the pattern now?

Exercise 2.6. Find the determinant.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 3 & -1 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 6 & 5 & 0 & 0 \\ 8 & 7 & 0 & 0 \\ 0 & 0 & 9 & 2 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

Example 2.7. Sometimes it's easy to find the determinant:

$$\det \begin{bmatrix} 3 & 91 & 23 \\ 0 & 2 & 57 \\ 0 & 0 & 11 \end{bmatrix} = 3 \cdot 2 \cdot 11 \qquad \det \begin{bmatrix} 5 & 91 & 23 & 38 \\ 0 & 3 & 57 & 9 \\ 0 & 0 & 17 & 64 \\ 0 & 0 & 0 & 10 \end{bmatrix} = 5 \cdot 3 \cdot 17 \cdot 10$$

This pattern holds for all (quadratic, of course) matrices on echelon form, because then there are only zeros below the main diagonal.

By using the properties of the determinant of matrices obtained by elementary row operations we get another method for computing the determinant: Use Gauss elimination to obtain a matrix on echelon form and keep track of how the operations change the determinant.

Example 2.8. Since

$$\begin{bmatrix} 3 & 91 & 23 \\ 0 & 20 & 570 \\ -6 & -180 & 22 \end{bmatrix} \sim \begin{bmatrix} 3 & 91 & 23 \\ 0 & 2 & 57 \\ -6 & -180 & 22 \end{bmatrix} \sim \begin{bmatrix} 3 & 91 & 23 \\ 0 & 2 & 57 \\ 0 & 2 & 68 \end{bmatrix} \sim \begin{bmatrix} 3 & 91 & 23 \\ 0 & 2 & 57 \\ 0 & 0 & 11 \end{bmatrix}$$

we get

$$\det \begin{bmatrix} 3 & 91 & 23 \\ 0 & 20 & 570 \\ -6 & -180 & 22 \end{bmatrix} = 10 \cdot \det \begin{bmatrix} 3 & 91 & 23 \\ 0 & 2 & 57 \\ 0 & 0 & 11 \end{bmatrix} = 10 \cdot 3 \cdot 2 \cdot 11$$

Since the determinant is non-zero the linear system

$$\begin{array}{llll} 3x_1 & +91x_2 & 23x_3 & = a \\ & 20x_2 & +570x_3 & = b \\ -6x_1 & -180x_2 & +22x_3 & = c \end{array}$$

has a solution no matter which numbers a, b and c we have (and there is only one solution).

Exercise 2.9. See ExercisesDet.

2.3. **Some matrices have an inverse.** First a warning. If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size we may multiply them in two ways: Either $\mathbf{A} \cdot \mathbf{B}$ or $\mathbf{B} \cdot \mathbf{A}$. These matrices are (usually) different!

Example 2.10.

$$\begin{bmatrix} 9 & 5 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 48 & 35 \\ 6 & 7 \end{bmatrix} \text{ while } \begin{bmatrix} 2 & 0 \\ 6 & 7 \end{bmatrix} \cdot \begin{bmatrix} 9 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 10 \\ 54 & 37 \end{bmatrix}$$

Definition 2.11. Let $\mathbf{I}_n$ denote the $(n \times n)$-matrix which has 1s on the diagonal and zeros elsewhere.

Example 2.12.

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{I}_1 = \begin{bmatrix} 1 \end{bmatrix}$$

Definition 2.13. Let $\mathbf{A}$ be a given $(n \times n)$-matrix. An *inverse* matrix to $\mathbf{A}$ is an $(n \times n)$-matrix $\mathbf{B}$ such that

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{I}_n \quad \text{and} \quad \mathbf{B} \cdot \mathbf{A} = \mathbf{I}_n$$

Example 2.14. Since

$$\begin{bmatrix} 7 & 11 \\ 5 & 8 \end{bmatrix} \cdot \begin{bmatrix} 8 & -11 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 8 & -11 \\ -5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 7 & 11 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

these two matrices are inverse to each other! (Do you see a pattern?)

Fact 1: If $\mathbf{A} \cdot \mathbf{B} = \mathbf{I}_n$ then $\mathbf{B} \cdot \mathbf{A} = \mathbf{I}_n$. So it's enough to check one of the equations.
Fact 2: There is at most one inverse to $\mathbf{A}$ (and there might be none!). If $\mathbf{A}$ has an inverse it's usually denoted $\mathbf{A}^{-1}$.

Exercise 2.15. Assume $\mathbf{A}$ is a (3×3)-matrix which has an inverse $\mathbf{A}^{-1}$ and $\det(\mathbf{A}) = 6$. Find $\det(\mathbf{A}^{-1})$.

2.4. **The determinant determines whether there is an inverse.**

Theorem 2.16. *The following is true for all square matrices* $\mathbf{A}$.

(1) *If* $\det(\mathbf{A}) \neq 0$ *then* $\mathbf{A}$ *has an inverse.*
(2) *If* $\mathbf{A}$ *has an inverse then* $\det(\mathbf{A}) \neq 0$.

Exercise 2.17. If
$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & 6 \\ 7 & 2 & 5 \end{bmatrix}$$
Does $\mathbf{A}$ have an inverse? Does $\mathbf{A}^2$ have an inverse?

Exercise 2.18. Assume $\mathbf{A}$ is a (3×3)-matrix with $\det(\mathbf{A}) = 5$. Put
$$\mathbf{B} = \begin{bmatrix} 2 & 0 & 0 \\ 59 & 3 & 1 \\ 476 & 2 & 0 \end{bmatrix}$$
Does $\mathbf{B}$ have an inverse? Find $\det(\mathbf{AB})$. Does $\mathbf{AB}$ have an inverse?

Exercise 2.19. Put
$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 9 & 2 \\ 0 & 3 & -5 & 1 \\ 0 & 7 & 4 & 3 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -9 & 7 & 0 & 0 \\ 3 & 8 & 1 & 0 \\ 11 & -5 & 0 & 0 \end{bmatrix}.$$
Does $\mathbf{A}$ have an inverse? Does $\mathbf{B}$ have an inverse? Does $\mathbf{AB}$ have an inverse?

Exercise 2.20. Assume $\mathbf{A}$ and $\mathbf{B}$ are (3×3)-matrices which both have inverses. Will $\mathbf{AB}$ necessarily have an inverse? What about $\mathbf{ABA}$?