Preparatory course in Linear Algebra

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Contents

References

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2. Knut Sydsæter and Peter J. Hammond, *Essential mathematics for economic analysis*, Prentice Hall, Harlow, 2008.

Lecture 1: Matrices and Matrix Algebra

Trond S. Gustavsen
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Reading. In this lecture we cover topics from Section 15.2 and 15.3 in [2]. Alternative reading is Section 8.3 in 7th edition of [1] (or Section 9.3 in the 5th or 6th edition).

1.1. **Matrices and Matrix Addition.** Matrices are rectangular arrays of numbers or symbols. Such arrangements of number and symbols occurs in many applications in economics, finance and statistics. Matrices allow for a compact notation and a more efficient handling since it is possible to do arithmetic directly on the matrices.

Definition 1. A matrix is a rectangular array of numbers considered as an entity.

We write down two matrices.

Example 2.

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ 6 & 5 \end{pmatrix}$$

Here A is a 2×2 matrix (two by two matrix) and B is a 3×2 matrix. We also say that A has size, order or dimension 2×2.

Definition 3. The sum of two matrices of the same order, is computed by adding the corresponding entries.

It is easy to see how this works in an example.

Example 4.

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ 6 & 5 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & -1 \\ 2 & 0 \end{pmatrix}$$

We can calculate the sum of A and C

$$A + C = \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 4 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 2+4 & 1+(-1) \\ 3+2 & 6+0 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 5 & 6 \end{pmatrix},$$

but the sum of A and B is not defined, since they do not have the same order.

Problem 1. Let

$$A = \begin{pmatrix} -2 & 1 \\ 4 & 2 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ -4 & -2 \\ -1 & 0 \end{pmatrix}.$$

Compute $A + B$.

A matrix can be multiplied by a number.

Definition 5. Let A be a matrix and let k be a real number. Then kA is calculated by multiplying each entry of A by k.

In an example this looks as follows:

Example 6. If

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix}$$

we get that

$$4A = \begin{pmatrix} 4 \cdot 2 & 4 \cdot 1 \\ 4 \cdot 3 & 4 \cdot 6 \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ 12 & 24 \end{pmatrix}.$$

If A and B are two matrices of the same size we define $A - B$ to be $A + (-1)B$. This means that matrices are subtracted by subtracting the corresponding entries.

Definition 7. We will write 0 for any matrix consisting of only zeros.

Example 8. The following are examples of zero matrices:

$$\underset{2\times 2}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} \quad \underset{2\times 4}{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \quad \underset{3\times 1}{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$

Problem 2. Let

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Compute $6A$, $(-1)B$ and $A - B$.

A *square matrix* is a matrix with the same number of rows and columns. In a square matrix, the elements (numbers or symbols) that sit in positions in the matrix where the row number and the column number are the same, constitute the *diagonal.*

Example 9. In the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

the elements 1, 5 and 9, are the elements on the diagonal.

Definition 10. The square matrix of order $n \times n$ that have only 1's on the diagonal and 0's elsewhere, is called the identity matrix and is denoted by I or I_n.

Example 11.

$$I_2 = \underset{2\times 2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$I_3 = \underset{3\times 3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

The following is *not* an identity matrix, since it is not a square matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Definition 12. Let A be an $n \times m$ matrix. The transpose of A, denoted A^T, is the $m \times n$ matrix obtained form A by interchanging the rows and columns in A.

For a 2×2 matrix, it looks as follows

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

We look at a 3×3 matrix in an example.

Example 13. Let

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Write down A^T.

Solution.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{pmatrix}.$$

Problem 3. Let

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 2 & -4 & 2 \end{pmatrix}.$$

Compute A^T and I_4^T.

The following rules apply:

Proposition 14. Let A, B and C be matrices of the same size (order), and let r and s be scalars (numbers). Then:

(1) $A + B = B + A$
(2) $(A + B) + C = A + (B + C)$
(3) $A + 0 = A$
(4) $r(A + B) = rA + rB$
(5) $(r + s)A = rA + sA$
(6) $r(sA) = (rs)A$

1.2. **Matrix Multiplication.** Matrix multiplication is slightly more complicated. We start by multiplying 2×2-matrices which are multiplied by the following rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

Example 15. Compute

$$\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix}.$$

Solution. We use the formula given above and obtain

$$\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 0 + (-2) \cdot 3 & 1 \cdot 1 + (-2) \cdot 1 \\ 0 \cdot 0 + 3 \cdot 3 & 0 \cdot 1 + 3 \cdot 1 \end{pmatrix} = \begin{pmatrix} -6 & -1 \\ 9 & 3 \end{pmatrix}.$$

Note the following pattern:

$$\begin{pmatrix} a & b \\ * & * \end{pmatrix}\begin{pmatrix} e & * \\ g & * \end{pmatrix} = \begin{pmatrix} ae+bg & * \\ * & * \end{pmatrix}$$

As a help, we can use the following diagram

$$\begin{array}{c|cc} & \begin{pmatrix} e & * \\ g & * \end{pmatrix} & = B \\ \hline A = \begin{pmatrix} a & b \\ * & * \end{pmatrix} & \begin{pmatrix} ae+bg & * \\ * & * \end{pmatrix} & = AB \end{array}$$

The same pattern can be use for other matrices, as in the following example:

Example 16. Let

$$A = \begin{pmatrix} 1 & 3 \\ -3 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Compute AB.

Solution.

$$\begin{array}{c|cc} & \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} & = B \\ \hline A = \begin{pmatrix} 1 & 3 \\ -3 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1\cdot 2+3\cdot 1 & 1\cdot 0+3\cdot 1 & 1\cdot 1+3\cdot 0 \\ (-3)\cdot 2+0\cdot 1 & (-3)\cdot 0+0\cdot 1 & (-3)\cdot 1+0\cdot 0 \\ 0\cdot 2+1\cdot 1 & 0\cdot 0+1\cdot 1 & 0\cdot 1+1\cdot 0 \end{pmatrix} & = AB \end{array}$$

We get that

$$AB = \begin{pmatrix} 5 & 3 & 1 \\ -6 & 0 & -3 \\ 1 & 1 & 0 \end{pmatrix}.$$

Note the orders of the matrices:

$$\underset{3\times 2}{A}\ \underset{2\times 3}{B} = \underset{3\times 3}{AB}.$$

You should now try to compute some matrices products on your own.

Problem 4. Let

$$A = \begin{pmatrix} 1 & 3 \\ -3 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 3 \\ -3 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Compute AB, BA, AC and BC if possible.

Matrix multiplication follows rules that are similar to the rules for multiplication of numbers.

Proposition 17. We have the following rules for matrix multiplication:

(1) $(AB)C = A(BC)$ (associative law)
(2) $A(B+C) = AB + AC$ (left distributive law)
(3) $(A+B)C = AC + BC$ (right distributive law)
(4) $IA = AI = A$
(5) $(AB)^T = B^T A^T$

Problem 5. Let

$$A = \begin{pmatrix} 1 & 3 \\ -3 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

Verify the proposition for these particular matrices.

1.3. **Homework.** Before you come to the next lecture, you should solve the following problems.

Problem 6. Compute $A+B$ and $5A$ when

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 & -3 \\ -3 & 0 \end{pmatrix}.$$

Problem 7. Compute $A+B$, $A-B$ and $3A-2B$ when

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 3 \\ -3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem 8. Compute AB and BA when

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 & -3 \\ -3 & 0 \end{pmatrix}.$$

Problem 9. Compute AB and BA, if possible, for the following:

(1) $A = \begin{pmatrix} 2 & 3 \\ 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(2) $A = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 & 0 \end{pmatrix}$

(3) $A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ 2 & 0 \\ 0 & 1 \end{pmatrix}$

Problem 10. The percentage that will vote for parties Left, Center and Right is given as follows:

	Left	Center	Right	No. of voters
Oslo	46 %	12 %	42 %	550 000
Akershus	40 %	12 %	48 %	500 000
Vestfold	46 %	10 %	44 %	253 000

Use matrix multiplication to compute the total number of voters for each party in the three regions.

Lecture 2: Matrix Algebra and Determinants

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Reading. In this lecture we cover topics from Section 15.4, 15.5, 15.7 and 16.1 in [2].

2.4. **Vectors.** A matrix with only one row is called a *row vector* and a matrix with only one column is called a *column vector*. We refer to both types as vectors. These are typically denoted by small bold letters and not capital letters. If a vector consists of n entries it is called an n-vector.

We may represent 2-vectors in a coordinate system.

Problem 11. Draw the vectors $\mathbf{a} = (1, 4)$ and $\mathbf{b} = (4, 1)$ in a coordinate system. Draw also $\mathbf{a} + \mathbf{b}$.

2.5. **More on Matrix Multiplication.** We will now see how to write a system of linear equations a matrix equation.

Example 18. Show that the system

$$\begin{aligned} 3x_1 + 4x_2 &= 5 \\ 7x_1 - 2x_2 &= 2 \end{aligned}$$

of linear equations can be written as

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{pmatrix} 3 & 4 \\ 7 & -2 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

Solution. We compute

$$A\mathbf{x} = \begin{pmatrix} 3 & 4 \\ 7 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + 4x_2 \\ 7x_1 - 2x_2 \end{pmatrix}$$

and we see that $A\mathbf{x} = \mathbf{b}$ if and only if $\begin{pmatrix} 3x_1 + 4x_2 \\ 7x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$. This is the same as saying that $3x_1 + 4x_2 = 5$ and $7x_1 - 2x_2 = 2$.

The advantage of writing a system on matrix form is that this compact form may be used even on very large systems of equations.

Problem 12. Write the following system of equations as $A\mathbf{x} = \mathbf{b}$:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_1 + 3x_2 + x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 1 \end{aligned}$$

Matrix notation can also be used to find the solution of a system of linear equations.

Problem 13. Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 2 & 5 & 3 \end{pmatrix} \text{ and } S = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$$

Compute $(SA)\mathbf{x}$ and use this to solve the system of linear equations in the previous problem.

This suggests the following definition.

Definition 19. Let A be any matrix. A matrix S is called an inverse of A if

$$AS = SA = I.$$

For a 2×2 matrix it is possible to give a formula for the inverse.

Problem 14. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and assume that $ad - bc \neq 0$. Show that

$$\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is an inverse of A.

An important fact is that the inverse matrix is unique.

Proposition 20. Let A be an $n \times n$ matrix. If A has an inverse, then it is unique.

Proof. Assume that both X and Y are inverses of A. Then

$$Y = IY = (XA)Y = X(AY) = XI = X.$$

□

Since the inverse of a matrix A is unique (if it exists) it is denoted by A^{-1}.

Problem 15. Find the inverse of

$$A = \begin{pmatrix} 3 & 4 \\ 7 & -2 \end{pmatrix}$$

and use this to solve

$$3x_1 + 4x_2 = 5$$
$$7x_1 - 2x_2 = 2$$

2.6. **Determinants.** We have already encountered determinants of 2×2 matrices.

Definition 21. The determinant of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is written

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

and is defined by

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Consider the following example.

Example 22. Compute the determinants

(a) $\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix}$

Solution.

(a) $\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 1 \cdot 3 - 4 \cdot 2 = -5$

(b) $\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = 2 \cdot 1 - 2 \cdot 3 = -4$

(c) $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix} = (a+b)^2 - (a-b)^2 = 4ab$

We shall later see how to compute the inverse of a 3×3 matrix by computing its so-called *cofactors.*

Definition 23. Let A be an 3×3 matrix. The cofactor A_{ij} is $(-1)^{i+j}$ times the determinant obtained by deleting row i and column j in A.

This definition will be generalized later.

Example 24. Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Compute the cofactor A_{12}.

Solution.

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} = (-1)(0 \cdot 1 - 2 \cdot (-1)) = -2$$

Problem 16. Compute some more cofactors of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}$$

The matrix

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

is called the *cofactor matrix* of A and denoted by $\operatorname{cof}(A)$. If $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}$ we compute that

$$\operatorname{cof}(A) = \begin{pmatrix} 1 & -2 & -2 \\ 0 & -3 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

Definition 25. The transpose of the cofactor matrix is called the *adjoint* matrix. In symbols

$$\operatorname{adj}(A) = \operatorname{cof}(A)^T.$$

The adjoint matrix has useful properties.

Problem 17. Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Find $\operatorname{adj}(A)$ and compute $A \operatorname{adj}(A)$ and $\operatorname{adj}(A)A$.

2.7. **Homework.** You should solve the following problems before the next lecture.

Problem 18. Draw the vectors $\mathbf{a} = (-1, 3)$ and $\mathbf{b} = (4, 2)$ in a coordinate system. Draw also $\mathbf{a} + \mathbf{b}$.

Problem 19. Compute the determinants
(a) $\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} a-b & a \\ a & a+b \end{vmatrix}$

Problem 20. Write

$$x_1 + x_2 = 1$$
$$x_1 - 2x_2 = 0$$

as

$$A\mathbf{x} = \mathbf{b}.$$

Find A^{-1} and use this to solve the system of equations.

Problem 21. Write the following system of equations as $A\mathbf{x} = \mathbf{b}$:

$$x_1 + 4x_2 + x_3 = 0$$
$$x_1 + 5x_2 + x_3 = 1$$
$$2x_1 + 9x_2 + 3x_3 = 1$$

Find the adjoint matrix $\text{adj}(A)$. Compute $\text{adj}(A)A$ and use this to solve the system of linear equation.

Lecture 3: Determinants and Inverse Matrices

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Reading. In this lecture we cover topics from Sections 16.2 through 16.6 in [2]. We will only cover parts of Sections 16.3 and 16.4

3.8. **Determinants of Order n and Cofactors.** In the previous lecture we saw how to compute determinants of two by two matrices and how to find cofactors in three by three matrices. Using cofactor expansion we can compute the determinants of three by three matrices.

Definition 26. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

be a 3×3 matrix. Then the determinant of A is given by

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

and this is called the cofactor expansion of $|A|$ along the first row.

We compute an example.

Example 27. Compute the determinant of A by cofactor expansion along the first row where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solution. We have

$$\begin{aligned} A &= 1 \cdot A_{11} + 2 \cdot A_{12} + 3 \cdot A_{13} \\ &= 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 2 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 3 \cdot (-1)^{1+3} \cdot \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} \\ &= (2 \cdot 1 - 1 \cdot 1) + 2 \cdot (-1) \cdot (3 \cdot 1 - 1 \cdot 1) + 3 \cdot (3 \cdot 1 - 1 \cdot 2) \\ &= 1 - 4 + 3 \\ &= 0 \end{aligned}$$

The determinant can also be computed by cofactor expansion along another row or along a column. This can reduce the numbers of calculations needed.

Problem 22. Compute the determinant of A by cofactor expansion along a suitable row where

(a) $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}$

Cofactor expansion can be used to compute determinants of any order. We compute an example with a four by four determinant.

Example 28. Compute the determinant of A by cofactor expansion along a suitable row where

$$A = \begin{pmatrix} 1 & -1 & -39 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ 2 & 1 & 40 & 2 \end{pmatrix}$$

Solution. By cofactor expansion along the second row we get

$$\begin{aligned} |A| &= 0 \cdot A_{21} + 0 \cdot A_{22} + 1 \cdot A_{23} + 0 \cdot A_{33} \\ &= 1 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 2 \end{vmatrix} \\ &= (-1) \cdot (1 \cdot (-1)^{2+1} \cdot \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} + 0 + 0) \\ &= (-1) \cdot ((-1)((-1) \cdot 2 - 1 \cdot 1) \\ &= (-1) \cdot ((-1)(-3)) \\ &= -3 \end{aligned}$$

Problem 23. Compute the determinant of A by cofactor expansion along a suitable row where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3.9. **Cramer's rule.** Cramer's rule is a useful way to solve a system of linear equations, in particular when we only need to know the value of one of the variables.

Proposition 29. Assume that A is an $n \times n$ matrix and assume that $|A| \neq 0$. Then

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution given as

$$x_i = \frac{|A_{\mathbf{b},i}|}{|A|} \text{ for } i = 1, 2, \ldots, n$$

where $A_{\mathbf{b},i}$ is obtained form by replacing the ith column with $\mathbf{b}$.

Let us consider an example.

Example 30. Write the following system of linear equations as $A\mathbf{x} = \mathbf{b}$ and solve it using Cramer's rule:

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= 4. \end{aligned}$$

Solution. We get

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

This gives

$$A_{\mathbf{b},1} = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix} \text{ and } A_{\mathbf{b},2} = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}.$$

Thus we get

$$x_1 = \frac{\begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{(1 \cdot (-1) - 4 \cdot 1)}{(1 \cdot (-1) - 1 \cdot 1)} = \frac{-5}{-2} = \frac{5}{2}$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{(1 \cdot 4 - 1 \cdot 1)}{(1 \cdot (-1) - 1 \cdot 1)} = \frac{3}{-2} = -\frac{3}{2}$$

We try another example.

Problem 24. Write the following system of linear equations as $A\mathbf{x} = \mathbf{b}$ and use Cramer's rule to find x_1:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_2 - 3x_3 &= 0 \\ x_1 + 4x_2 - 3x_3 &= 0 \end{aligned}$$

3.10. **The Inverse Matrix.** In Lecture 2 we learned about the inverse matrix. Now that we have learned about determinants, we can give a formula for the inverse matrix.

Proposition 31. Assume that A is an $n \times n$ matrix. Then A is invertible if and only if $|A| \neq 0$. If $|A| \neq 0$, then

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A).$$

We look at some examples.

Problem 25. Determine if the following matrix is invertible:

$$\begin{pmatrix} 12 & -3 & 4 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Problem 26. Determine if the given matrix invertible and if so find its inverse.

(a) $A = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ -2 & 1 & 1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solutions to Lecture 1: Matrices and Matrix Algebra

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Problem 6. Compute $A+B$ and $5A$ when
$$A=\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \text{ and } B=\begin{pmatrix} -2 & -3 \\ -3 & 0 \end{pmatrix}.$$

Solution. $A+B=\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}+\begin{pmatrix} -2 & -3 \\ -3 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix}$

$5A=5\cdot\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 10 & 15 \\ 5 & 5 \end{pmatrix}$

Problem 7. Compute $A+B$, $A-B$ and $3A-2B$ when
$$A=\begin{pmatrix} 2 & 3 \\ 2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B=\begin{pmatrix} 1 & 3 \\ -3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Solution. $A+B=\begin{pmatrix} 2 & 3 \\ 2 & 0 \\ 0 & 1 \end{pmatrix}+\begin{pmatrix} 1 & 3 \\ -3 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 3 & 6 \\ -1 & 0 \\ 0 & 2 \end{pmatrix}$

$A-B=\begin{pmatrix} 2 & 3 \\ 2 & 0 \\ 0 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 3 \\ -3 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 5 & 0 \\ 0 & 0 \end{pmatrix}$

$3A-2B=3\begin{pmatrix} 2 & 3 \\ 2 & 0 \\ 0 & 1 \end{pmatrix}-2\begin{pmatrix} 1 & 3 \\ -3 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 4 & 3 \\ 12 & 0 \\ 0 & 1 \end{pmatrix}$

Problem 8. Compute AB and BA when
$$A=\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \text{ and } B=\begin{pmatrix} -2 & -3 \\ -3 & 0 \end{pmatrix}.$$

Solution. $AB=\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & -3 \\ -3 & 0 \end{pmatrix}=\begin{pmatrix} -13 & -6 \\ -5 & -3 \end{pmatrix}$

$BA=\begin{pmatrix} -2 & -3 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} -7 & -9 \\ -6 & -9 \end{pmatrix}$

Problem 9. Compute AB and BA, if possible, for the following:

(1) $A = \begin{pmatrix} 2 & 3 \\ 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(2) $A = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 & 0 \end{pmatrix}$

(3) $A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ 2 & 0 \\ 0 & 1 \end{pmatrix}$

Solution. (1)

$$AB = \begin{pmatrix} 2 & 3 \\ 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 & 3 \\ 6 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 9 \\ 0 & 1 \end{pmatrix}$$

(2)

$$AB = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 2 & 0 \\ 6 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = 8$$

(3)

AB is not defined.

$$BA = \begin{pmatrix} 2 & 3 \\ 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 9 \\ 4 & 6 \\ 1 & 1 \end{pmatrix}$$

Problem 10. The percentage that will vote for parties Left, Center and Right is given as follows:

	Left	Center	Right	No. of voters
Oslo	46 %	12 %	42 %	550 000
Akershus	40 %	12 %	48 %	500 000
Vestfold	46 %	10 %	44 %	253 000

Use matrix multiplication to compute the total number of voters for each party in the three regions.

Solution. $\frac{1}{100} \begin{pmatrix} 46 & 40 & 46 \\ 12 & 12 & 10 \\ 42 & 48 & 44 \end{pmatrix} \begin{pmatrix} 550000 \\ 500000 \\ 253000 \end{pmatrix} = \begin{pmatrix} 569\,380 \\ 151\,300 \\ 582\,320 \end{pmatrix}$

Left gets 389 380, Center gets 151 300 and Right gets 366 320.

Solutions to Lecture 2: Matrix Algebra and Determinants

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August 7th, 2009

Problem 18. Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Find $\operatorname{adj}(A)$ and compute $A \operatorname{adj}(A)$ and $\operatorname{adj}(A)A$.

Solution.

$$\operatorname{adj}(A) = \begin{pmatrix} 1 & 0 & -2 \\ -2 & -3 & 1 \\ -2 & 0 & 1 \end{pmatrix}$$

$$A \operatorname{adj}(A) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ -2 & -3 & 1 \\ -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

$$\operatorname{adj}(A)A = \begin{pmatrix} 1 & 0 & -2 \\ -2 & -3 & 1 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Problem 19. Compute the determinants

(a) $\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} a-b & a \\ a & a+b \end{vmatrix}$

Solution. (a) $\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} = -14$ (b) $\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = 1$ (c) $\begin{vmatrix} a-b & a \\ a & a+b \end{vmatrix} = -b^2$

Problem 20. Write

$$x_1 + x_2 = 1$$
$$x_1 - 2x_2 = 0$$

as

$$A\mathbf{x} = \mathbf{b}.$$

Find A^{-1} and use this to solve the system of equations.

Solution. $A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$A^{-1} = \frac{1}{(1)(-2)-(1)(1)} \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^{-1}\mathbf{b} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

Problem 21. Write the following system of equations as $A\mathbf{x} = \mathbf{b}$:

$$x_1 + 4x_2 + x_3 = 0$$
$$x_1 + 5x_2 + x_3 = 1$$
$$2x_1 + 9x_2 + 3x_3 = 1$$

Find the adjoint matrix $\text{adj}(A)$. Compute $\text{adj}(A)A$ and use this to solve the system of linear equation.

Solution. $A = \begin{pmatrix} 1 & 4 & 1 \\ 1 & 5 & 1 \\ 2 & 9 & 3 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\text{adj}(A) = \begin{pmatrix} 6 & -3 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$

$\text{adj}(A)A = \begin{pmatrix} 6 & -3 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ 1 & 5 & 1 \\ 2 & 9 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 & -3 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$