$$x_1 = \frac{d_1 + \beta d_2}{1 - \alpha\beta\gamma}, \qquad x_2 = \frac{\alpha\gamma d_1 + d_2}{1 - \alpha\beta\gamma}, \qquad x_3 = \frac{\alpha d_1 + \alpha\beta d_2}{1 - \alpha\beta\gamma} \tag{**}$$

Clearly, this solution for (x_1, x_2, x_3) only makes sense when $\alpha\beta\gamma < 1$. In fact, if $\alpha\beta\gamma \geq 1$, it is impossible for this economy to meet any positive final demands for fish and timber—production in the economy is too inefficient.

PROBLEMS FOR SECTION 15.1

1. Decide which of the following single equations in the variables x, y, z, and w are linear and which are not. (In (f), a and b are nonnegative constants.)

   (a) $3x - y - z - w = 50$
   (b) $\sqrt{3}x + 8xy - z + w = 0$
   (c) $3(x + y - z) = 4(x - 2y + 3z)$
   (d) $3.33x - 4y + \frac{800}{3}z = 3$
   (e) $(x - y)^2 + 3z - w = -3$
   (f) $2a^2x - \sqrt{b}y + (2 + \sqrt{a})z = b^2$

SM 2. Write system (1) out in full when $n = m = 4$ and $a_{ij} = 1$ for all $i \neq j$, while $a_{ii} = 0$ for $i = 1, 2, 3, 4$. Sum the four equations to derive a simple equation for $\sum_{i=1}^{4} x_i$, then solve the whole system.

3. Write down the system of equations (1) when $n = m = 3$ and $a_{ij} = i + j$ for $i, j = 1, 2, 3$, while $b_j = j$ for $j = 1, 2, 3$.

4. In Example 2 let $\alpha = 1/2$, $\beta = 1/4$, $\gamma = 2$, $d_1 = 100$, and $d_2 = 80$. Write down system (∗) in this case and find the solution of the system. Confirm the results by using the general formulas in (∗∗).

5. Consider a collection of n individuals, each of whom owns a definite quantity of m different commodities. Let a_{ij} be the number of units of commodity i owned by individual j, where $i = 1, 2, \ldots, m$, while $j = 1, 2, \ldots, n$.

   (a) What does the list $(a_{1j}, a_{2j}, \ldots, a_{mj})$ represent?

   (b) Explain in words what $a_{11} + a_{12} + \cdots + a_{1n}$ and $a_{i1} + a_{i2} + \cdots + a_{in}$ express.

   (c) Let p_i denote the price per unit of commodity i $(i = 1, 2, \ldots, m)$. What is the total value of the commodities owned by individual j?

SM 6. T. Haavelmo devised a model of the U.S. economy for the years 1929–1941 based on the following equations:

   (i) $c = 0.712y + 95.05$ (ii) $s = 0.158(c + x) - 34.30$
   (iii) $y = c + x - s$ (iv) $x = 93.53$

   Here x denotes total investment, y is disposable income, s is the total saving by firms, and c is total consumption. Write the system of equations in the form (1) when the variables appear in the order x, y, s, and c. Then find the solution of the system.

The last row represents the equation $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 2a - 3b + c$. The system therefore has solutions only if $2a - 3b + c = 0$. In this case the last row has only zeros, and we continue using elementary operations till we end up with the following matrix:

$$\begin{pmatrix} 1 & 0 & -1/3 & 4/3 & \frac{1}{3}(a+2b) \\ 0 & 1 & -2/3 & -1/3 & \frac{1}{3}(b-a) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and thus} \quad \begin{cases} x_1 \qquad - \frac{1}{3}x_3 + \frac{4}{3}x_4 = \frac{1}{3}(a+2b) \\ \qquad x_2 - \frac{2}{3}x_3 - \frac{1}{3}x_4 = \frac{1}{3}(b-a) \end{cases}$$

Here x_3 and x_4 can be freely chosen. Once they have been chosen, however, x_1 and x_2 are uniquely determined linear functions of $s = x_3$ and $t = x_4$:

$$\begin{aligned} x_1 &= \tfrac{1}{3}(a+2b) + \tfrac{1}{3}s - \tfrac{4}{3}t \\ x_2 &= \tfrac{1}{3}(b-a) \;\; + \tfrac{2}{3}s + \tfrac{1}{3}t \end{aligned} \qquad (s \text{ and } t \text{ arbitrary real numbers, } 2a - 3b + c = 0)$$

For $2a - 3b + c \neq 0$ the given system is inconsistent, so has no solutions.

PROBLEMS FOR SECTION 15.6

1. Solve the following systems by Gaussian elimination.

(a) $\begin{aligned} x_1 + x_2 &= 3 \\ 3x_1 + 5x_2 &= 5 \end{aligned}$ (b) $\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_1 - x_2 + x_3 &= 5 \\ 2x_1 + 3x_2 - x_3 &= 1 \end{aligned}$ (c) $\begin{aligned} 2x_1 - 3x_2 + x_3 &= 0 \\ x_1 + x_2 - x_3 &= 0 \end{aligned}$

2. Use Gaussian elimination to discuss what are the possible solutions of the following system for different values of a and b:

$$\begin{aligned} x + y - z &= 1 \\ x - y + 2z &= 2 \\ x + 2y + az &= b \end{aligned}$$

SM 3. Find the values of c for which the system

$$\begin{aligned} 2w + x + 4y + 3z &= 1 \\ w + 3x + 2y - z &= 3c \\ w + x + 2y + z &= c^2 \end{aligned}$$

has a solution, and find the complete solution for these values of c.

SM 4. Consider the two systems of equations:

(a) $\begin{aligned} ax + y + (a+1)z &= b_1 \\ x + 2y + z &= b_2 \\ 3x + 4y + 7z &= b_3 \end{aligned}$ (b) $\begin{aligned} \tfrac{3}{4}x + y + \tfrac{7}{4}z &= b_1 \\ x + 2y + z &= b_2 \\ 3x + 4y + 7z &= b_3 \end{aligned}$

Find the values of a for which (a) has a unique solution, and find all solutions to system (b).