FORK1003
Exercises for Lecture 2

August 4, 2015

1 Matrices and Matrix Operations

1.1 Matrix Defined

Exercise 1.1 (Matrix coordinates)**.**

$$\begin{bmatrix} 5 & -23 & 18 & 39 & -30 \\ 6 & 8 & -5 & 13 & 7 \\ 1 & -9 & -12 & 64 & -15 \\ -4 & -11 & 46 & 81 & -2 \end{bmatrix}$$

For the following coordinates, give the corresponding entry in the matrix above:

(a) (2, 4)

(b) (3, 1)

(c) (1, 4)

(d) (4, 2)

1.2 Addition and Scalar Multiplication

Exercise 1.2. Define the matrices

$$A = \begin{bmatrix} 3 & -2 \\ 6 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ -8 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & -4 & 3 \\ -5 & 0 & -2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 8 & -7 \end{bmatrix}.$$

Calculate each of the following expressions, if it is defined:

(a) $A + B$

(b) $C - D$

(c) $B - D$

(d) $3B$

(e) $2C - 3D$

1.3 Matrix Multiplication

Exercise 1.3. Compute the following dot products:

(a) $\begin{bmatrix} 1 & 3 & -2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} 16 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} a & b & c & d & e \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 3 \end{bmatrix}$

Exercise 1.4.

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 & 1 \\ -3 & -1 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & -4 & 0 \\ -5 & 0 & 2 \\ 1 & 3 & -1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 2 \\ -3 & 0 \\ 8 & -7 \end{bmatrix}, \qquad E = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}, \qquad F = \begin{bmatrix} 4 & 2 & -3 \end{bmatrix}$$

Compute each of the following expression, if it is defined:

(a) AB

(b) BA

(c) CB

(d) BC

(e) CE

(f) CF

(g) FC

(h) EF

(i) FE

1.5 Transpose

Exercise 1.5.

$$A = \begin{bmatrix} 1 & 4 \\ -2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 0 & -1 \\ 1 & 2 & -4 \end{bmatrix},$$

$$C = \begin{bmatrix} 5 & -1 & -2 \\ 0 & 3 & 1 \\ 1 & -4 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}$$

Compute the following expressions:

(a) A^T

(b) B^T

(c) CD^T

(d) $B^T A$

(e) $A^T B$

1.7 Square Matrices

Exercise 1.6. Let

$$A = \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$$

Compute A^2.

Exercise 1.7. This question is about diagonal matrices.

(a) Calculate $\begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^2$

(b) If $A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$, can you write A^k in a general form, for any positive integer k?

2 Inverse Matrices

2.1 Briefly on Determinants

Exercise 2.1. By calculating the determinant, determine whether the following 2×2 matrices are invertible:

(a) $A = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 3 \\ 2 & -3 \end{bmatrix}$ (c) $C = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

2.2 Finding the Inverse

Exercise 2.2. Calculate the inverses of the following matrices using row reduction. Check each answer by seeing if $AA^{-1} = I_n$.

(a) $A = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 11 \\ -1 & 0 & -8 \end{bmatrix}$ (c) $C = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 2 & 8 & 6 \\ 3 & 7 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

2.3 Extra: Rearranging Matrix Equations

Exercise 2.3. Suppose $n \times n$ matrices A, B, C and D are all invertible and satisfy the equation

$$A = B(D - 3I_n)C.$$

Solve for D in terms of A, B and C. (That is, rearrange to get D by itself on one side of the equation)

3 Linear Systems as Matrix Equations

Exercise 3.1. Write the following linear systems as matrix equations:

(a) $$\begin{cases} \quad\quad x_2 - 3x_3 + \ \ x_4 = 2 \\ -x_1 \quad\quad + \ \ x_3 + 6x_4 = -1 \\ 9x_1 - 7x_2 + \ \ x_3 \quad\quad = 13 \end{cases}$$

(b) $$\begin{cases} x_1 - x_2 = 3 \\ x_1 + x_2 = 6 \end{cases}$$

(c) $$\begin{cases} \quad\quad x_2 \quad\quad - \ \ x_4 = 1 \\ 3x_1 - \ \ 4x_2 + 5x_3 \quad\quad = 3 \\ \quad\quad 15x_2 + 2x_3 \quad\quad = -20 \\ x_1 + \ \ x_2 - 3x_3 - 4x_4 = 0 \end{cases}$$

3.2 Solving Linear Systems Through Matrix Equations

Exercise 3.2. Solve the following linear systems by inverting their coefficient matrices:

(a) $$\begin{cases} 2x_1 - \ \ x_2 = 2 \\ 4x_1 + 2x_2 = -4 \end{cases}$$

(b) $$\begin{cases} 3x_1 - \ \ x_2 + \ \ x_3 = 4 \\ 2x_1 \quad\quad + \ \ x_3 = 1 \\ \quad\quad - 4x_2 - 4x_3 = -2 \end{cases}$$

4 Linear Systems as Linear Combinations of Columns

Exercise 4.1. Write out the following linear system as a linear combination of columns:

$$\begin{cases} x_1 \quad\quad + 3x_3 - \ \ x_4 = 2 \\ \quad\quad 5x_2 - \ \ x_3 - 2x_4 = 0 \\ 6x_1 - 7x_2 + \ \ x_3 \quad\quad = -3 \end{cases}$$

4.2 Linear Combinations

Exercise 4.2.

$$\underline{\mathbf{v}}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \underline{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \qquad \underline{\mathbf{v}}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

When possible, write each of the following vectors as a linear combination of the vectors above:

$$\text{(a)} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(d)} \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix} \qquad \text{(e)} \begin{bmatrix} 6 \\ 3 \\ 1 \\ 0 \end{bmatrix} \qquad \text{(f)} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

4.4 Solving Linear Systems

Exercise 4.3. Determine whether the following sets of vectors span the given vector space:

(a) $\mathbb{R}^2$; $\quad \underline{\mathbf{v}}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \underline{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(b) $\mathbb{R}^3$; $\quad \underline{\mathbf{v}}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \underline{\mathbf{v}}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

(c) $\mathbb{R}^3$; $\quad \underline{\mathbf{v}}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{\mathbf{v}}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \underline{\mathbf{v}}_3 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

(d) $\mathbb{R}^4$; $\quad \underline{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \underline{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{\mathbf{v}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \underline{\mathbf{v}}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Exercise 4.4. By finding the span of the column vectors, show that the linear system

$$\begin{cases} 3x_1 \phantom{{}+x_2} + x_3 = b_1 \\ x_1 + x_2 \phantom{{}-x_3} = b_2 \\ 2x_1 + x_2 - x_3 = b_3 \end{cases}$$

has a solution for every $\underline{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.