FORK1003
Exercises for Lecture 3

August 6, 2015

1 Introduction to Determinants

Exercise 1.1. Calculate the determinants of the following 2×2 matrices:

(a) $A = \begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$ (b) $B = \begin{bmatrix} 0 & 13 \\ -2 & 1 \end{bmatrix}$ (c) $C = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

2 Clever Trick for 3×3 Determinants

Exercise 2.1. Using the "drawing lines" method, calculate the determinants of the following matrices:

(a) $A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 0 & -3 \\ 0 & 10 & 1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 4 & 6 \\ -1 & -2 & -3 \\ 5 & 3 & -4 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 0 \\ -1 & 2 & 3 \end{bmatrix}$

3 Cofactor Expansion

3.1 Minors and Cofactors

Exercise 3.1. For a matrix A, denote by A_{ij} the matrix obtained from A by removing the ith row and the jth column. For

$$A = \begin{bmatrix} 7 & -5 & 2 & 4 \\ -2 & 0 & 3 & 1 \\ -1 & 2 & 0 & 6 \\ 3 & -2 & -5 & 1 \end{bmatrix},$$

write the following matrices

(a) A_{11} (b) A_{23} (c) A_{14}

Exercise 3.2. Let A be the matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 4 & 6 & 0 \\ -1 & -2 & 5 \end{bmatrix}$$

Calculate the following minors and cofactors:

(a) M_{21}

(b) M_{33}

(c) C_{11}

(d) C_{32}

3.2 Cofactor Expansion

Exercise 3.3. Calculate the determinants of the following matrices by cofactor expansion along a suitable row or column:

(a) $A = \begin{bmatrix} 4 & -2 & -1 \\ 0 & 1 & 3 \\ 2 & -3 & 1 \end{bmatrix}$

(b) $B = \begin{bmatrix} -2 & 3 & -1 \\ 0 & 5 & 0 \\ 7 & -2 & 1 \end{bmatrix}$

(c) $C = \begin{bmatrix} 2 & -3 & 0 & 5 \\ 28 & 13 & 2 & -6 \\ 1 & -1 & 0 & 3 \\ 2 & 3 & 0 & -4 \end{bmatrix}$

(d) $D = \begin{bmatrix} 3 & 2 & -5 & 2 \\ -2 & 1 & -1 & 4 \\ -3 & -1 & -6 & 2 \\ 0 & -4 & 0 & 0 \end{bmatrix}$

4 Determinants by Row Reduction

4.1 Determinants and Elementary Row Operations

Exercise 4.1. Suppose we have a 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

with the matrix $|A|$. Write the determinants of the following matrices in terms of $|A|$:

(a) $B = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$

(b) $C = \begin{bmatrix} a & b & c \\ d & e & f \\ -2g & -2h & -2i \end{bmatrix}$

(c) $D = \begin{bmatrix} a+3d & b+3e & c+3f \\ d & e & f \\ g & h & i \end{bmatrix}$

(d) $E = \begin{bmatrix} g & h & i \\ a & b & c \\ d & e & f \end{bmatrix}$

(e) $F = \begin{bmatrix} a & b & c \\ 2d-3a & 2e-3b & 2f-3c \\ g & h & i \end{bmatrix}$

(f) $G = \begin{bmatrix} 2a & 2b & 2c \\ g & h & i \\ -d & -e & -f \end{bmatrix}$

Exercise 4.2. Using row reduction and the formula for determinants of upper-diagonal matrices, calculate determinants for the following matrices:

(a) $A = \begin{bmatrix} 6 & 2 & -4 \\ 3 & -1 & 5 \\ 0 & 1 & 3 \end{bmatrix}$

(b) $B = \begin{bmatrix} -5 & 2 & 1 \\ -1 & 3 & 2 \\ 2 & 7 & 4 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & 4 & 1 & 2 \\ 2 & 2 & 3 & -5 \\ 6 & -4 & 0 & 2 \end{bmatrix}$

(d) $D = \begin{bmatrix} 3 & 2 & 1 & -3 \\ 4 & 6 & -1 & -2 \\ 0 & 1 & 2 & 1 \\ 1 & 4 & 3 & 1 \end{bmatrix}$

4.3 Combining Cofactor Expansion and Row Reduction

Exercise 4.3. Calculate the determinants of the following matrices through a combination of row reduction and cofactor expansion:

(a) $A = \begin{bmatrix} 4 & 2 & -4 & 6 \\ 1 & 0 & 2 & -2 \\ 2 & 0 & 0 & 0 \\ 6 & 3 & -7 & 1 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 5 & 7 & 3 & -4 \\ 0 & 1 & 0 & 3 & -1 \\ 0 & -2 & 5 & 1 & 2 \\ 0 & 3 & 0 & -1 & 3 \\ -1 & -5 & 1 & 2 & 1 \end{bmatrix}$

5 The Adjugate Matrix and Inverses

Exercise 5.1. Find the inverses of the following matrices by calculating their adjugate matrix:

(a) $A = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & -2 \\ 0 & -3 & 4 \end{bmatrix}$

(b) $B = \begin{bmatrix} 6 & 2 & -1 \\ 0 & 3 & -2 \\ -1 & 1 & -2 \end{bmatrix}$

6 Cramer's Rule

Exercise 6.1. Use Cramer's rule to solve for the following linear systems:

(a)

$$\begin{cases} -x_1 + x_2 + x_3 = 1 \\ x_1 + 2x_2 = 0 \\ x_1 + 2x_2 + 3x_3 = 0. \end{cases}$$

(b)

$$\begin{cases} x_1 + x_2 + 3x_3 = 3 \\ -4x_1 + x_2 - 3x_3 = 2 \\ 5x_1 + 2x_2 + 2x_3 = -1. \end{cases}$$