1

Cofactor expansion

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -7 & 5 \\ 1 & -1 & 6 \end{bmatrix}.$$

Find C_{31}.

~~A_{31}~~

~~Ex~~ 1: Find A_{31}: $= \begin{bmatrix} 0 & 1 \\ -7 & 5 \end{bmatrix}$

2: Find $M_{31} = \det(A_{31}) = 0 \cdot 5 - 1 \cdot (-7)$
$= 7.$

3: $C_{31} = (-1)^{3+1} \cdot M_{31} = 1 \cdot M_{31} = 1 \cdot 7 = 7.$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 1 & 4 \\ 2 & -1 & 3 \end{bmatrix}.$$

1: Cofactor expand along row 1.

So A_{11}, A_{12}, A_{13}

$A_{11} = \begin{pmatrix} 1 & 4 \\ -1 & 3 \end{pmatrix}$, $A_{12} = \begin{pmatrix} -3 & 4 \\ 2 & 3 \end{pmatrix}$, $A_{13} = \begin{pmatrix} -3 & 1 \\ 2 & -1 \end{pmatrix}$.

2. Find minors:

$M_{11} = 1 \cdot 3 - (-1) \cdot 4 = 7$. $M_{12} = -3 \cdot 3 - 4 \cdot 2 = -17$.

$M_{13} = (-3) \cdot (-1) - 2 \cdot 1 = \underline{1}$.

3. Find cofactors: $C_{11} = (-1)^{1+1} M_{11} = M_{11} = \underline{\underline{7}}$

$C_{12} = (-1)^{1+2} M_{12} = -M_{12} = \underline{\underline{17}}$.

$C_{13} = (-1)^{1+3} M_{13} = M_{13} = \underline{\underline{1}}$

② (cont...)

$$\det(A) = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + a_{13} \cdot C_{13}$$
$$= 1 \cdot C_{11} + 2 \cdot C_{12} + 0 \cdot C_{13}$$
$$= 1 \cdot 7 + 2 \cdot 17 + 0 \cdot 1 = 41 = \det(A)$$

Choose any line:

$$\begin{bmatrix} 1 & 20 & 32 \\ 0 & 3 & -4 \\ 0 & 5 & -2 \end{bmatrix}$$

4x4 #

$$A = \begin{bmatrix} 1 & 6 & -3 & 13 \\ 0 & 2 & -3 & 8 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

1: Cofactor expand along column 1.

$$\det(A) = \underbrace{1 \cdot (-1)^{1+1} \det(A_{11})}_{a_{11} \cdot C_{11}} + 0 \ldots + 0.$$

$$A_{11} = \begin{pmatrix} 2 & -3 & 8 \\ 0 & 3 & -2 \\ 0 & 2 & -1 \end{pmatrix}, \quad \det(A_{11}) = M_{11}$$

$$(-1)^{1+1} M_{11} = C_{11}.$$

③ (cont...)

$$B = \begin{pmatrix} 2 & -3 & 8 \\ 0 & 3 & -2 \\ 0 & 2 & -1 \end{pmatrix}$$

$$\det(B) = 2 \cdot C^*_{11} + 0 \cdot C^*_{21} + 0 \cdot C^*_{31}$$

$$= 2 \cdot C^*_{11},$$

$$B_{11} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}. \quad M^*_{11} = \det(B_{11}) = 3 \cdot (-1) - (-2) \cdot 2$$

$$= 1.$$

$$C^*_{11} = (-1)^{1+1} \cdot M^*_{11} = M^*_{11} = \underline{1}.$$

Going back to 4x4:

$$\det(A) = 1 \cdot \det(B) = 2 \cdot C^*_{11} = \underline{\underline{2 \cdot 1}}.$$

4 Row reduction:

$$A = \begin{bmatrix} 2 & 8 & 4 & -6 \\ -2 & -5 & 2 & 11 \\ -1 & 2 & 3 & 13 \\ 1 & 4 & 2 & -5 \end{bmatrix} \qquad (\det(A) =) |A|$$

$R1 \to \frac{1}{2} R1$

$$\begin{bmatrix} 1 & 4 & 2 & -3 \\ -2 & -5 & 2 & 11 \\ -1 & 2 & 3 & 13 \\ 1 & 4 & 2 & -5 \end{bmatrix} \qquad \frac{1}{2}|A|$$

$R2 \to R2 + 2R1$
$R3 \to R3 + R1$
$R4 \to R4 - R1$

$$\begin{bmatrix} 1 & 4 & 2 & -3 \\ -2+2=0 & 3 & 6 & 5 \\ 0 & 6 & 5 & 10 \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad \frac{1}{2}|A|$$

$R3 \to R3 - 2 \cdot R2$

$$\begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & 3 & 6 & 5 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Done.

$$\frac{1}{2}|A| = 1 \cdot 3 \cdot (-7) \cdot (-2) = 42$$

So $|A| = \det(A) = 2 \cdot 42 = 84$

5 Cofactors + row reduction

$$A = \begin{bmatrix} 3 & 0 & -4 & 2 \\ -1 & 0 & 3 & 5 \\ 2 & 5 & 1 & -2 \\ -3 & 0 & 2 & 6 \end{bmatrix}. \text{ Find det.}$$

1. Cofactor expand on col. 2.

$a_{32} = 5$.

$$B = A_{32} = \begin{bmatrix} 3 & -4 & 2 \\ -1 & 3 & 5 \\ -3 & 2 & 6 \end{bmatrix}. \quad C_{32} = (-1)^{3+2} \cdot M_{32} = -M_{32}.$$

So.

$$\det(A) = 5 \cdot C_{32} = -5 M_{32} = -5 \det(B).$$

$$B = \begin{bmatrix} 3 & -4 & 2 \\ -1 & 3 & 5 \\ -3 & 2 & 6 \end{bmatrix} \quad |B|$$

$R3 \to$ ~~R3+R1~~ $R3+R1$

$$\begin{bmatrix} 3 & -4 & 2 \\ -1 & 3 & 5 \\ 0 & -2 & 8 \end{bmatrix} \quad |B|$$

$R1 \to R1 + 3 \cdot R2$

$$\begin{bmatrix} 0 & 5 & 17 \\ -1 & 3 & 5 \\ 0 & -2 & 8 \end{bmatrix} \quad |B|$$

6 (cont...)

Cofactor expand along col. 1.

$\det(B) = -1 \cdot C_{21} = -1 \cdot (-1)^{2+1} \cdot M_{21} = M_{21}$

$M_{21} = \det\left(\begin{bmatrix} 5 & 17 \\ -2 & 8 \end{bmatrix}\right) = 5 \cdot 8 - 17 \cdot (-2)$

$= 40 + 34 = 74 = \det(B)$

So $\det(A) = -5 \cdot \det(B) = -5 \cdot 74 = -370$

(7) Adjugate:

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 1 \\ 5 & -1 & -1 \end{bmatrix}$$

1. Find cofactor matrix:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$$

$$= \begin{bmatrix} +\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix} \\ -\begin{vmatrix} 3 & -1 \\ -1 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 5 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} \\ +\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -2+1 & -(-1-5) & -1-10 \\ -(-3-1) & -1+5 & -(-1-15) \\ 3+2 & -(1+1) & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & 6 & -11 \\ 4 & 4 & 16 \\ 5 & -2 & -1 \end{bmatrix}$$

(8) (cont...)

2: Take transpose:

$\mathrm{adj}(A) = C^T$. $C = \begin{pmatrix} -1 & 6 & -11 \\ 4 & 4 & 16 \\ 5 & -2 & -1 \end{pmatrix}$

$$\mathrm{adj}(A) = C^T = \begin{bmatrix} -1 & 4 & 5 \\ 6 & 4 & -2 \\ -11 & 16 & -1 \end{bmatrix}.$$

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)}.$$

3: Find $\det(A)$: $A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 1 \\ 5 & -1 & -1 \end{bmatrix}$

Expand along Row 1:

$$|A| = 1 \cdot C_{11} + 3 \cdot C_{12} - 1 \cdot C_{13}$$

$$= 1 \cdot M_{11} - 3 \cdot M_{12} - 1 \cdot M_{13}$$

$$= 1 \cdot \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} - 3 \cdot \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix}$$

$$= 1(-2+1) - 3 \cdot (-1-5) - 1 \cdot (-1-10)$$

$$= -1 + 18 + 11 = \underline{\underline{28}}.$$

$$A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{adj}(A) = \frac{1}{28} \begin{bmatrix} -1 & 4 & 5 \\ 6 & 4 & -2 \\ -11 & 16 & -1 \end{bmatrix}$$

9 Cramer:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 2 \\ -1 & 2 & 1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 4 \\ 10 \\ -4 \end{bmatrix}$$

~~1: Find det(A)~~

$$A_1 = \begin{bmatrix} 4 & 2 & 1 \\ 10 & 0 & 2 \\ -4 & 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 10 & 2 \\ -1 & -4 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 10 \\ -1 & 2 & -4 \end{bmatrix}$$

1: Find det(A):

$$\det(A) = 3 \cdot \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix}$$

$$= 3(-4) - 2(1+2) + 1(1 \cdot 2) = -12 - 6 + 2$$

$$= -16.$$

2: Solve for x_1: Find $\det(A_1)$:

$$\det(A_1) = 4 \cdot \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 10 & 2 \\ -4 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 10 & 0 \\ -4 & 2 \end{vmatrix}$$

$$= 4 \cdot (0-4) - 2(10+8) + 1(20)$$

$$= -16 - 36 + 20$$

$$= -32.$$

3: Use Cramer:

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-32}{-16} = \underline{\underline{2}}.$$