LECTURE 4

FORK 1003 BI

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MATHEMATICS

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Plan:

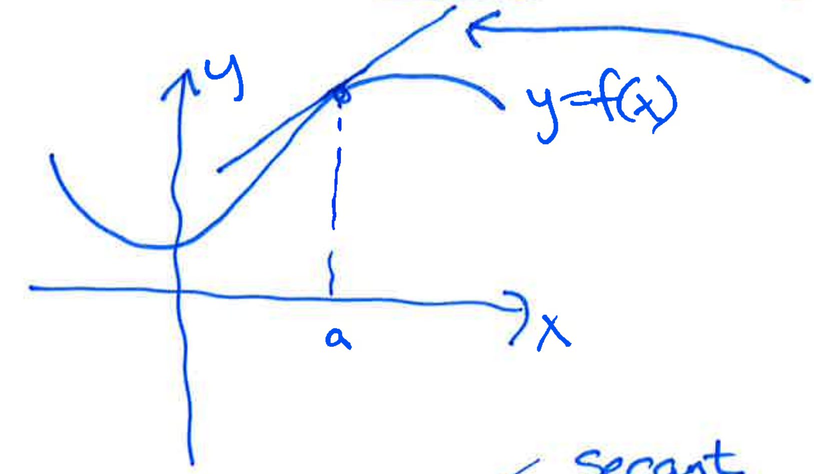
1. Functions and the derivative
2. Computing the derivative
3. Exponential functions and logarithms
4. Higher derivatives
5. Applications
6. Functions in two variables and partial derivatives
7. Higher derivatives and Hesse-matrices
8. Applications

postponed to Lecture 5.

Textbook:

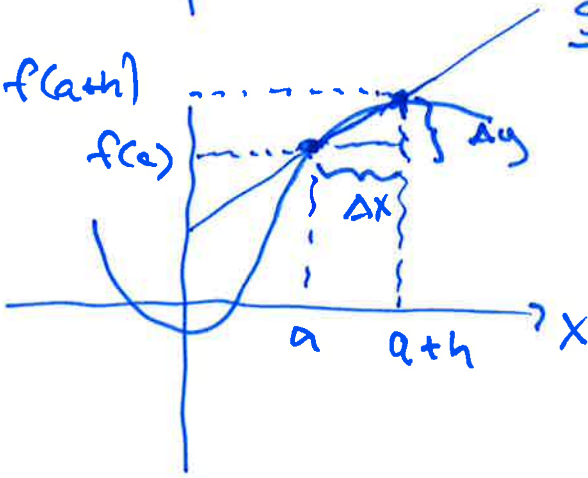
[ME] Simon, Blume "Maths for Economists"
Ch. 2-5, 13-14

(1) Functions and the derivative



tangent to the graph of f at x=a

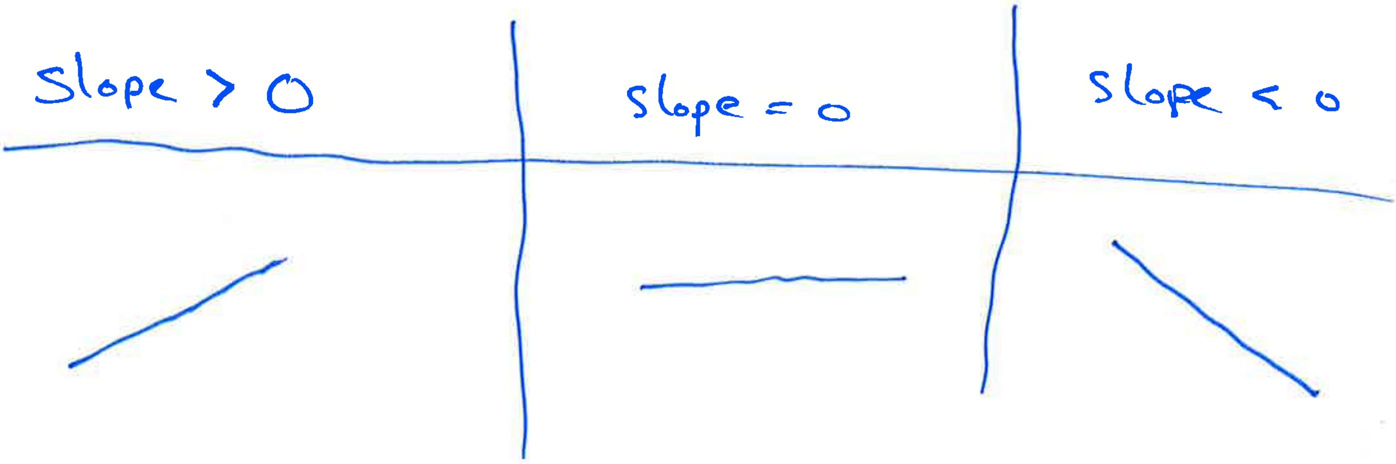
$f'(a)$ = slope of this tangent



Secant

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{f(a+h)-f(a)}{h}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$



② Computing the derivative

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As a function: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

slope of the tangent line
at any pt. x

Ex: $f(x) = x^3 - 3x + 2$

$f'(x) = \underline{\underline{3x^2 - 3}}$ $\quad f'(1) = 3 \cdot 1^2 - 3 = \underline{0}$

Derivation rules: Write $f'(x) = (f(x))' = \frac{df}{dx}$

(1) $(x^n)' = n \cdot x^{n-1}$ for all n

(2) $\left.\begin{array}{l} (u+v)' = u' + v' \\ (u-v)' = u' - v' \end{array}\right\}$ for all expressions $u = u(x)$, $v = v(x)$

(3) $(c \cdot u)' = c \cdot u'$ for all $u = u(x)$, c = constant

(4) $(u \cdot v)' = u' \cdot v + u \cdot v'$

(5) $\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$

for all expr. $u = u(x)$, $v = v(x)$

(6) $f(x) = f(u)$, where $u = u(x)$

$\Downarrow$

$f'(x) = f'(u(x)) \cdot u'(x)$ **Chain rule**

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

Ex:

$$(\sqrt{x})' = (x^{1/2})' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$(\sqrt{x^2+1})' = \frac{1}{2\sqrt{u}}\cdot(x^2+1)' = \frac{1}{2\sqrt{u}}\cdot 2x = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$$

$\sqrt{u}$, where $u = x^2+1$

Chain rule.

$$\left(\frac{x+1}{x^2+4}\right)' = \left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2} = \frac{1\cdot(x^2+4)-(x+1)\cdot 2x}{(x^2+4)^2}$$

$$= \frac{x^2+4-2x^2-2x}{(x^2+4)^2} = \frac{-x^2-2x+4}{(x^2+4)^2}$$

Quotient rule

$$\left(x\cdot\sqrt{x+3}\right)' = (u\cdot v)' = u'v+uv'$$

product rule

$$= 1\cdot\sqrt{x+3} + x\cdot\frac{1}{2\sqrt{x+3}}\cdot 1$$

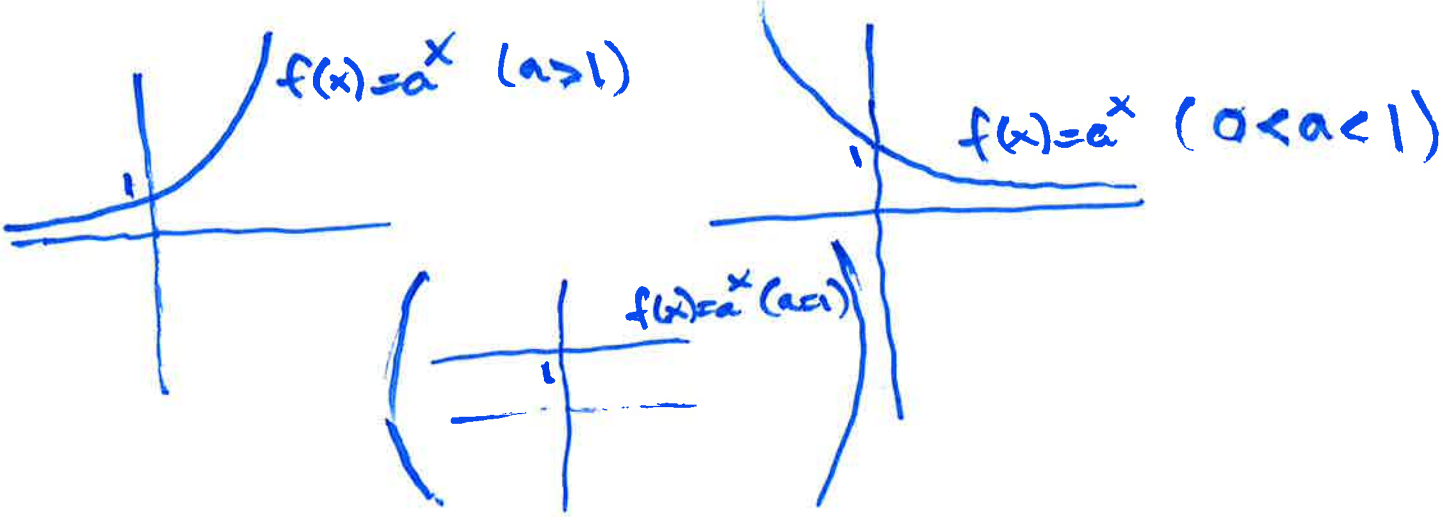
$$= \sqrt{x+3} + \frac{x}{2\sqrt{x+3}}$$

$$= \frac{2(x+3)}{2\sqrt{x+3}} + \frac{x}{2\sqrt{x+3}} = \frac{3x+6}{2\sqrt{x+3}}$$

3 Exponential functions and logarithms

Exponential functions:

$f(x) = a^x$ ($a>0$ fixed number (base), $a \neq 1$)



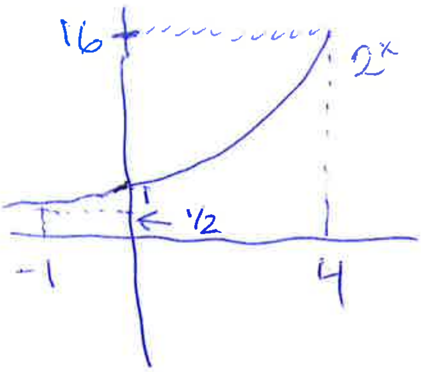
Logarithms:

$f(x) = \log_a(x)$ = inverse fn. of a^x

This means:

$\log_a(x) = y \iff a^y = x$

Ex: $\log_2(16) =$ soln. of $\boxed{2^x = 16}$



$= 4$

$\log_2(1/2) = \log_2(2^{-1}) = -1$

Note: $\log_a(x)$ is defined for $x > 0$

In other words, for any base $a > 0$ with $a \neq 1$:

$f(x) = a^x$ has
- $D_f = \mathbb{R}$ (all real numbers)
- $V_f = (0, \infty)$

$f(x) = \log_a(x)$ has
- $D_f = (0, \infty)$
- $V_f = \mathbb{R}$

Euler number:

$e = 2{,}71828\ldots \longrightarrow$ $f(x) = e^x$

$f(x) = \log_e(x) = \ln(x)$

natural logarithm

Rules for powers and logarithms:

i) $a^x \cdot a^y = a^{x+y}$

ii) $a^x / a^y = a^{x-y}$

iii) $(a^x)^y = a^{x \cdot y}$

i) $\log_a(xy) = \log_a(x) + \log_a(y)$

ii) $\log_a(x/y) = \log_a(x) - \log_a(y)$

iii) $\log_a(x^y) = y \cdot \log_a(x)$

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<u>In particular:</u>

$(a=e)$

$$\begin{cases} \ln(xy) = \ln(x) + \ln(y) \\ \ln(x/y) = \ln(x) - \ln(y) \\ \ln(x^y) = y \cdot \ln(x) \end{cases}$$

<u>Ex:</u> $2^x = 12$

$\swarrow \log_2(-)$ $\qquad$ $\searrow \ln(-)$

$$\log_2(2^x) = \log_2(12)$$

$$x = \underline{\log_2(12)}$$

$$= \log_2(\underbrace{2 \cdot 2}\cdot 3)$$

$$= \log_2(2^2) + \log_2(3)$$

$$= \underline{2 + \log_2(3)}$$

$$\ln(2^x) = \ln(12)$$

$$x \cdot \ln(2) = \ln(12)$$

$$x = \frac{\ln(12)}{\ln(2)}$$

$$= \frac{\ln(2^2) + \ln(3)}{\ln(2)}$$

$$= \frac{2\ln(2) + \ln(3)}{\ln(2)}$$

$$= \underline{2 + \frac{\ln 3}{\ln 2}}$$

$$= 3{,}58\ldots$$

$$\simeq \underline{3{,}58}$$

$$\boxed{\underline{\text{Fact:}} \quad \log_a(x) = \frac{\ln(x)}{\ln(a)}}$$

Derivatives:

$f(x) = a^x$

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x (a^h - 1)}{h} = a^x \cdot \boxed{\lim_{h \to 0} \frac{a^h - 1}{h}}$$

$$= a^x \cdot \ln(a)$$

Conclusion: Derivative of the exponential fn.

i) $(a^x)' = a^x \cdot \ln(a)$

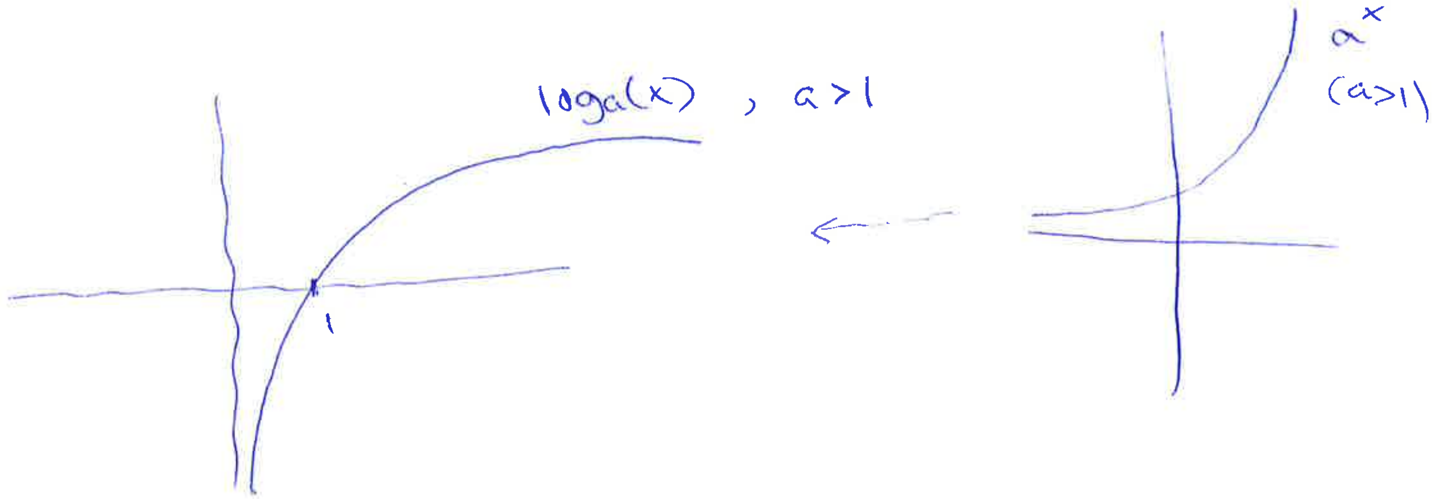
$\boxed{\text{ii) } (e^x)' = e^x}$ $\leftarrow$ special case $a = e$

Derivation of logarithms:

i) $(\log_a(x))' = \frac{1}{x} \cdot \frac{1}{\ln(a)}$

$\boxed{\text{ii) } (\ln(x))' = \frac{1}{x}}$ $\leftarrow$ $a = e$

Graph of $\log_a(x)$:



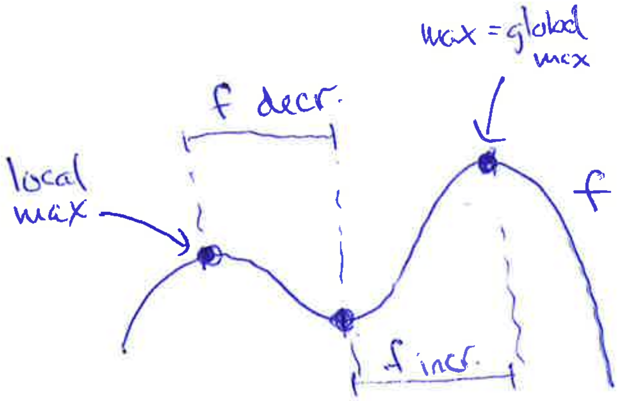
(4) Higher derivatives

$f''(x)$ = the derivative of $f'(x)$

Ex: $f(x) = x^3 - 3x + 2$

$f'(x) = \underline{3x^2 - 3}$

$f''(x) = \underline{6x}$



(5) Applications

a) Increasing / decreasing:

f increasing in [a,b] if $f'(x) \geq 0$ for all x in (a,b)

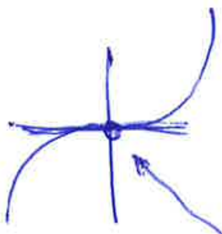
" decreasing " $f'(x) \leq 0$ "

b) Max/min: $\boxed{x=a \text{ stationary pt. if } f'(a)=0}$

max/min $\Rightarrow$ local max/min $\Rightarrow$ stationary pt.

stationary pt $\Rightarrow$ either $\begin{cases} \text{local max} \\ \text{local min} \\ \text{saddle pt.} \end{cases}$

Ex:



$f'(x) = 3x^2$

$f'(x)=0$:

$3x^2=0$

$\underline{x=0}$ ← stationary pts

Saddle pt.

$f'(x)$: $+$ $\overset{x=0}{\circ}$ $+$

$f''(x) = 6x$

$f''(0) = \underline{0}$

Procedure for finding max/min = optimization

i) Compute $f'(x)$.

ii) Solve $f'(x)=0$ ⟶ Stationary pts.
(can be zero, one, many...)

iii) Classify each stationary pt.
as local max, local min, saddle pt.

iv) For each local max/min, try to determine
if it is also $\underline{\text{max/min}}$ = global max/min.

Classification of stationary pts: (iii)

$\underline{x=a}$:
Stationary pt.

Alt A: Sign diagram for $f'(x)$

Ex: $f'(x)$ . . . . $-$ $-$ $\overset{x=a}{\circ}$ $+$

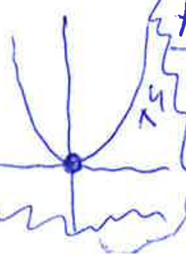
Ex: $f(x)=x^4$

$f'(x) = 4x^3 = 0$

$\underline{x=0}$ stat. pts.

$f''(x) = 12x^2$

$f''(0) = \underline{0}$



Alt B: Second derivative test

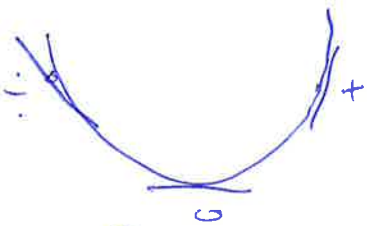
$f''(a)>0$: $x=a$ local min

$f''(a)<0$: $x=a$ local max

c) Convex / concave

f is convex if $f''(x) \geq 0$ for all x (in D_f)

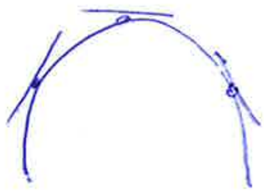
" concave " $f''(x) \leq 0$ — " — (in D_f)



f convex

$f''(x) \geq 0$

$f'(x)$ increasing



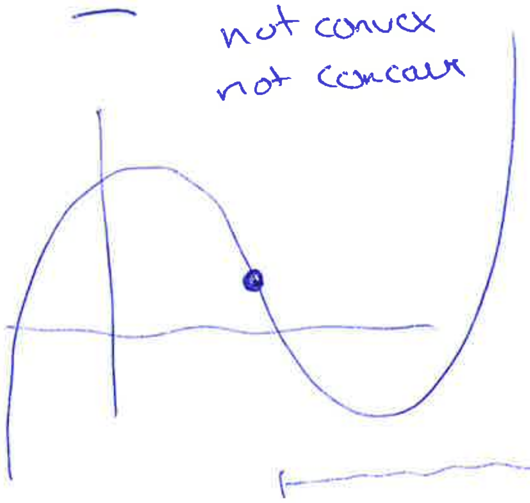
f concave

$f''(x) \leq 0$

$f'(x)$ decreasing

Ex:

not convex
not concave

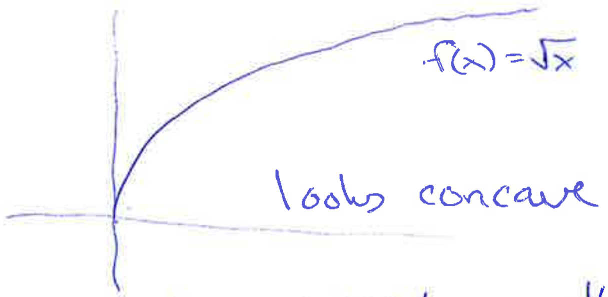


$f''(x) < 0$

$f''(x) \geq 0$

$\frac{1}{x\sqrt{x}} = \frac{1}{x \cdot x^{0,5}} = \frac{1}{x^{1,5}}$

Ex: $f(x) = \sqrt{x}$, $x \geq 0$



$f'(x) = \left(x^{1/2}\right)' = \frac{1}{2} x^{-1/2}$

$f''(x) = \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot x^{-3/2}$

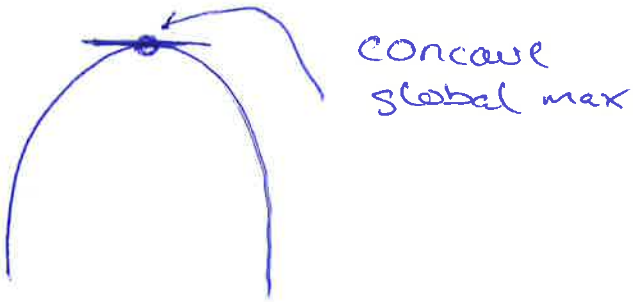
$= \frac{-1}{4 x \sqrt{x}} < 0$

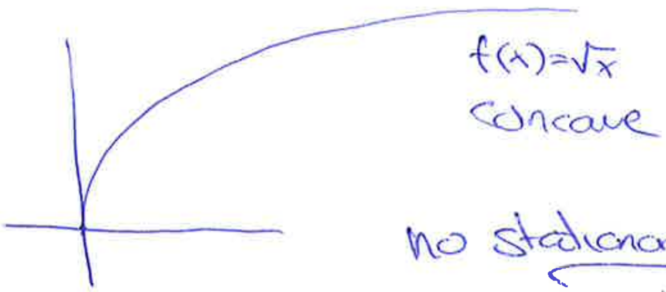
f is concave.

Fact:

If f is convex, then any stationary pt is a global min.

If f is concave, then any stationary pt is a global max.





no stationary pts $\Rightarrow$ no global max.