LECTURE 5

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FORK1003 MATHEMATICS

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Plan:

3. Functions in two variables and partial derivation
4. The Hesse matrix
5. ~~Applications~~ (next time)

} from Lecture 4

1. Integration
2. Integration by parts, substitution and integration of rational functions

} Integration

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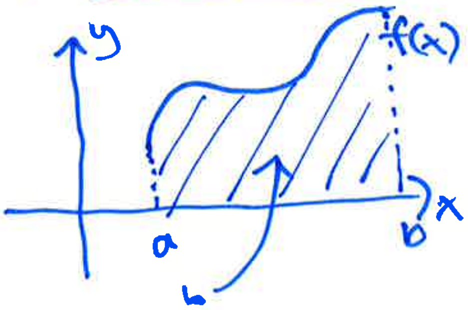
1 Integration:



$$\text{Area} = \int_a^b f(x)\,dx$$

definite integral

$$= \left[F(x) + C \right]_a^b$$

$$= F(b) - F(a)$$

Defn: An anti-derivative of $f(x)$ to be a function $F(x)$ s.t.

$$F'(x) = f(x)$$

Ex: $f(x) = 2x$

$F(x) = x^2,\ x^2+1,\ x^2+C$ are antiderivatives of $f(x)$ (C const.)

General antiderivative:

$$\underbrace{\int f(x)\,dx}_{\text{indefinite integral}} = x^2 + C$$

$f(x)$: fn. we look for an antiderivative of

$\int \ldots dx$: notation, x is the variable we diff. w.r.t.

Facts:

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① If $F(x)$ is one antiderivative of $f(x)$ on $[a,b]$, then the general antiderivative is $F(x)+C$.

$$F'(x) = f(x) \Rightarrow \int f(x)\,dx = F(x) + C$$

② If $f(x)$ is continuous on $[a,b]$, then there is an antiderivative $F(x)$.

Ex: $f(x) = e^{-x^2}$ $F(x)$ exists, but cannot be "expressed as a formula in x" (not elementary fn.)

Computing integrals:

① $\int x^n\,dx = \frac{1}{n+1}\cdot x^{n+1} + C, \quad n \neq -1$

② $\left.\begin{aligned} \int (u+v)\,dx &= \int u\,dx + \int v\,dx \\ \int (u-v)\,dx &= \int u\,dx - \int v\,dx \end{aligned}\right\}$ $u=u(x),\ v=v(x)$

③ $\int c\cdot u(x)\,dx = c\cdot \int u(x)\,dx$ $\quad u=u(x),\ c$ const.

④ $\int \frac{1}{x}\,dx = \ln|x| + C$

⑤ $\int e^x\,dx = e^x + C$

$\int a^x\,dx = \frac{1}{\ln(a)}\cdot a^x + C$

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Ex:

$$\int x^3\,dx = \frac{1}{4}x^4 + C \qquad n=3,\ \text{power rule } (1)$$

$$\left(\tfrac{1}{4}x^4\right)' = \frac{1}{\cancel{4}}\cdot\cancel{4}x^3 = x^3$$

$$\int x^4 - 3x^2 + x - 7\,dx =$$

$$\frac{1}{5}\cdot x^5 - \cancel{3}\cdot\frac{1}{\cancel{3}}x^3 + \frac{1}{2}\cdot x^2 - 7x + C$$

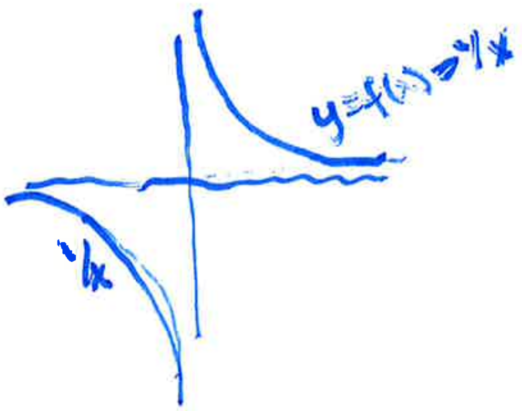
$$= \underline{\frac{1}{5}x^5 - x^3 + \frac{1}{2}x^2 - 7x + C}$$

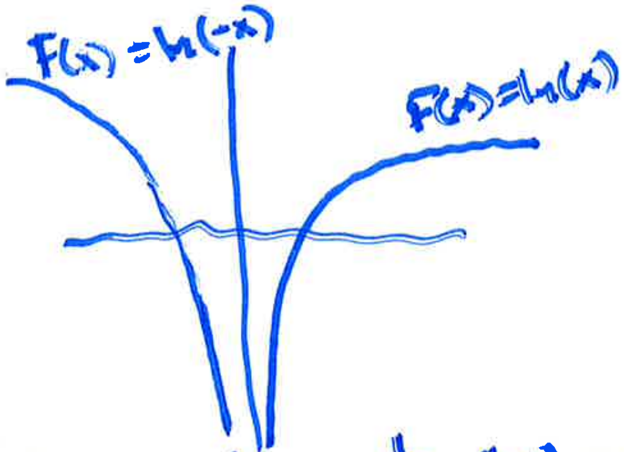
$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{1}{3/2}\cdot x^{3/2} + C$$

$$= \underline{\frac{2}{3}\cdot x^{3/2} + C}$$

$$\int \frac{1}{x}\,dx = \ln(x) + C$$

$$(4)\quad \int \frac{1}{x}\,dx = \ln|x| + C \quad ?$$





$$(\ln(-x))' = \frac{1}{u}\cdot(-1) = \frac{1}{(-x)}\cdot(-1)$$

$$= \frac{1}{x}$$

General antiderivative:

$$F(x) = \left\{\begin{array}{ll} \ln(x) & , x>0 \\ \ln(-x) & , x<0 \end{array}\right\} = \ln|x|$$

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{1}{-1}\cdot x^{-1} + C$$

$$= -\frac{1}{x} + C$$

② Integration techniques

- Integration by parts (products)
- Substitution (composite fn's)
- Integration of rational functions

Integration by parts: $\int uv'\,dx = uv - \int u'v\,dx$

Product rule:

$(u\cdot v)' = u'v + u\cdot v'$

$uv' = (uv)' - u'v$

$\int uv'dx = uv - \int u'v\,dx$

Ex: $\int x\cdot e^x\,dx = xe^x - \int 1\cdot e^x\,dx$

$u = x \qquad v = e^x$

$u' = 1 \qquad v' = e^x$

$= xe^x - \int e^x\,dx = \underline{xe^x - e^x + C}$

$\int xe^x\,dx = \frac{1}{2}x^2e^x - \int \frac{1}{2}x^2e^x\,dx$

$u = e^x \qquad v = \frac{1}{2}x^2$

$u' = e^x \qquad v' = x$

$\int a\cdot v'\,dx \qquad uv - \int u'v\,dx$

Ex: $\int \ln x\,dx = \int 1\cdot \ln x\,dx = x\cdot \ln x - \int \frac{1}{x}\cdot x\,dx$

$u = \ln x \qquad v = x$

$u' = 1/x \qquad v' = 1$

$= x\cdot \ln x - \int 1\,dx$

$= \underline{x\ln x - x + C}$

Substitutions

$$\boxed{du = u' dx}$$

<u>Ex:</u> $\int e^{1-x}\,dx = \frac{1}{-1} e^{1-x} + C = \underline{-e^{1-x} + C}$

$\boxed{e^{1-x} = e^{u},\ u = 1-x}$

$u' = -1$

<u>Alt</u> $\int e^{1-x}\,dx = \int e^{u} \cdot \frac{du}{(-1)} = \int -e^{u}\,du$

want to transform this into an integral in u

$\boxed{\begin{matrix} u = 1-x \\ du = -1 \cdot dx \end{matrix}} \rightarrow dx = \frac{du}{(-1)}$

$\boxed{du = u' dx}$

$$= -e^{u} + C = \underline{-e^{1-x} + C}$$

<u>Ex:</u> $\int x \cdot \sqrt{x^2+1}\,dx = \int \cancel{x} \sqrt{u} \cdot \frac{du}{2\cancel{x}}$

$\boxed{\begin{matrix} u = x^2+1 \\ du = 2x \cdot dx \end{matrix}} \rightarrow dx = \frac{du}{2x}$

$$= \int \frac{1}{2} \sqrt{u}\,du = \frac{1}{2} \int u^{1/2} du = \frac{1}{\cancel{2}} \left(\frac{\cancel{2}}{3} \cdot u^{3/2} \right) + C$$

$$= \frac{1}{3} u^{3/2} + C = \frac{1}{3} u\sqrt{u} + C$$

$$= \underline{\frac{1}{3}(x^2+1)\sqrt{x^2+1} + C}$$

Integration of fractions: (Rational fn's)

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Ex: $\displaystyle\int \frac{x^2}{x+1}\,dx$

$\parallel$

$\displaystyle\int \left(x-1+\frac{1}{x+1}\right)dx$

$\displaystyle = \frac{1}{2}x^2 - x + \int \frac{1}{x+1}\,dx$

$$= \frac{1}{2}x^2 - x + \ln|x+1| + C$$

Polynomial div.:

$$x^2 : x+1 = x-1$$
$$-(x^2+x)$$
$$-x$$
$$-(-x-1)$$
$$1$$

$\Downarrow$

$$\frac{x^2}{x+1} = x-1+\frac{1}{x+1}$$

$\displaystyle\int \frac{1}{x+1}\,dx = \int \frac{1}{u}\,du$

$\boxed{\begin{array}{c} u = x+1 \\ du = 1\cdot dx \end{array}}$

$= \ln|u| + C$

$= \ln|x+1| + C$

Ex:

$\displaystyle\int \frac{3}{x^2-3x+2}\,dx$

$\displaystyle = \int \frac{3}{(x-2)\cdot(x-1)}\,dx$

$\displaystyle = \int \left(\frac{A}{x-2} + \frac{B}{x-1}\right)dx$

$\displaystyle = \int \frac{3}{x-2} + \frac{-3}{x-1}\,dx$

$$= 3\cdot\ln|x-2| - 3\ln|x-1| + C$$

$\displaystyle\frac{3}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \quad \Big|\cdot(x-2)(x-1)$

$\displaystyle 3 = \frac{A\,\cancel{(x-2)}(x-1)}{\cancel{x-2}} + \frac{B\cdot(x-2)\cancel{(x-1)}}{\cancel{x-1}}$

$3 = A\cdot(x-1) + B(x-2)$

$3 = \underbrace{(A+B)}_{=0}x + \underbrace{(-A-2B)}_{=+3}$

$A+B=0$
$-A-2B=3$

$A = 3$
$B = -3$

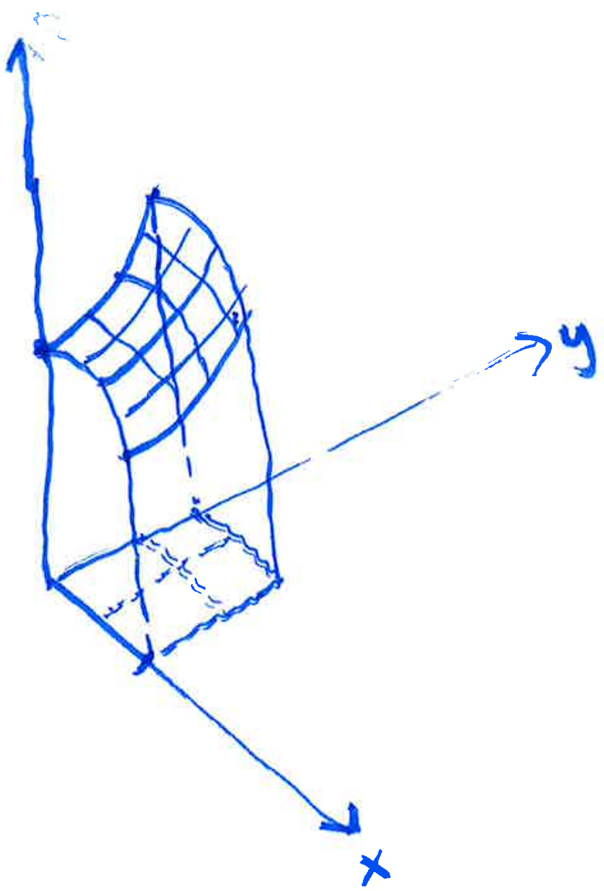
(3) Functions in two variables and partial derivation

Ex: $f(x,y) = x^2+y^2+1$

__Graph:__ $z = f(x,y) = x^2+y^2+1$

$(1,1,f(1,1)) = (1,1,3)$

pt. on the graph of f



Partial derivatives:

$f(x,y)$: fn. in two variables

Partial derivatives:

$$f'_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$ slope of tangent in x-direction

$$f'_y = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$ slope of tangent in y-direction

Computing partial derivatives

Ex: $f(x,y) = x^2+y^2+1$

$$f'_x(x,y) = 2x+0+0 = \underline{2x}$$

$$f'_y(x,y) = 0+2y+0 = \underline{2y}$$

$$f(x,y) = x^3-3xy+y^3$$

$u'_x \cdot v + u \cdot v'_x$

$$f'_x = 3x^2 - 3(xy)'_x = 3x^2-3(1\cdot y + x\cdot 0)$$

$$= \underline{3x^2-3y}$$

$$f'_y = 0 - 3x\cdot 1 + 3y^2 = \underline{-3x+3y^2}$$

(2) Hessian matrix

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Ex: $f(x,y) = x^3 - 3xy + y^3$

$$\left.\begin{aligned} f'_x &= 3x^2 - 3y \\ f'_y &= -3x + 3y^2 \end{aligned}\right\} \quad \begin{aligned} f''_{xx} &= 6x & f''_{xy} &= -3 \\ f''_{yx} &= -3 & f''_{yy} &= 6y \end{aligned}$$

Hessian matrix:

$$H(f)(x,y) = \begin{pmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{yx}(x,y) & f''_{yy}(x,y) \end{pmatrix}$$

$f = x^3 - 3xy + y^3$: $\quad H(f) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$

Fact:

C^2 functions

For all "nice" functions, $f''_{xy} = f''_{yx}$.
This means that $H(f)$ is symmetric.

A is symmetric if $A^T = A$.